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TRIAL CALCULATION OF A HULL FORM OF DECREASED WAVE RESISTANCE BY THE METHOD OF STEEP DESCENT

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ABSTRACT

Since the design of ships always involves compromise between various desired characteristics, it is necessary to find ways to develop hull forms in which reduced wave-making resistance is incorporated in that compromise. Some way is needed to take hull forms which fulfill the other requirements placed on them and then modify them in such a way as to reduce their wave-making resistance. The method of steep descent appears suitable for this purpose. A hull form generated by fifty-five sources placed on the center line plane of a hull in five horizontal rows is modified using this method so that its wave-making resistance is decreased 45 percent while the shape of the waterline plane and the hull volume are, to first order, held constant.

INTRODUCTION AND DESCRIPTION OF THE METHOD

The design of ship hulls is controlled by a great many factors other than the need to reduce the wave-making resistance. Therefore a most desirable technique for reducing wave-making resistance is to start with a hull form which meets these other requirements and then modify it continuously in such a way as to reduce the wave-making resistance. When the change has gone as far as possible without making the hull incompatible with the other requirements placed on it, then the change must be stopped.

In 1936, Hogner[1] published a paper which took the first step toward developing a method of this sort. He promised a sequel which apparently was never published. It turns out, however, that the necessary mathematical machinery to finish the development of his method has been brought forth in the past few years, and is known as the method of steep descent. This is not to be confused with the saddle-point method of the same name which has been used for a long time. The method of steep descent referred to here requires that we describe the resistance of the ship as a function of a finite number of variables. The hull form is also described in terms of these variables. Then if we consider each of these variables to represent a coordinate in a finite-dimensional vector space, we may find the gradient of the resistance in this vector space. If, next, we change our defining variables in a direction as nearly parallel as possible to this gradient vector, then we will decrease the resistance as rapidly as possible for a given amount of motion through the vector space.

Note: Because of corrections made to the tabulated functions on which this paper's calculations were based, it was necessary to recalculate the results which were presented at the Seminar. Although the first set of calculations were performed with a slide rule, those presented in this version of the paper were performed on an IBM 1620 computer.

If it is desired to hold some quantity fixed -- the hull volume, for example -- then it is possible to constrain the trajectory through the vector space so that it is orthogonal to the gradient of that quantity. This can be done by rotating coordinates so that the vector space is spanned by a set of basic vectors, one of them parallel to the gradient of the quantity to be held constant and all the others orthogonal to it. Then the vector parallel to the gradient of the fixed quantity is held constant, but the other vectors are allowed to vary. A detailed justification of this technique is included in the author's paper published earlier this year.[2]

In that paper a single elementary result of the method of steep descent was also displayed, a near-optimum infinitely deep strut. That result will be used here as the starting point of a calculation in which the shape of the hull will be permitted to vary with depth. Instead of an infinitely deep strut, however, this calculation will start with fifty-five sources equally spaced in five rows over a rectangle on the centerline plane with a ratio of depth to length $T/L \approx .05$. The initial strengths of the sources will be approximately equivalent to the distribution of dipole moment in the strut but with finite rather than infinite depth. The strength of the top row of sources and the longitudinal first moment of the source strengths will be held constant. To first order, this is equivalent to holding the volume and the shape of the waterline plane constant throughout the calculation. Finally, it will be explicitly assumed that the fore-and-aft symmetry of the strut will persist, and the calculation will be simplified to take advantage of this fact. The Froude number $f = .316$ of the strut calculation will be used in order to keep the conditions of the problem the same. This Froude number corresponds to the product $K_0 L = 10$ and so simplifies work with a slide rule.

The five rows of sources will be placed at depths .005L, .015L, .025L, .035L, and .045L . Then the original ship can be described by the following set of sources on the centerline plane, equally spaced in depth.

	Station										
Depth	0	1	2	3	4	5	6	7	8	9	10
.005L	.66	.10	.25	.20	.14	.00	-.14	-.20	-.25	-.10	-.66
.015L	.66	.10	.25	.20	.14	.00	-.14	-.20	-.25	-.10	-.66
.025L	.66	.10	.25	.20	.14	.00	-.14	-.20	-.25	-.10	-.66
.035L	.66	.10	.25	.20	.14	.00	-.14	-.20	-.25	-.10	-.66
.045L	.66	.10	.25	.20	.14	.00	-.14	-.20	-.25	-.10	-.66

Table. 1. Initial Source Strengths on the Centerline Plane.

The requirement that the sum of the source strengths be zero is met by this set of sources. It will persist throughout the calculation, and so closure of the streamlines will be retained.

It can be shown(2) that the resistance of a set of sources arranged on the centerline plane is given by the following equation:

$$R = 16\pi\rho K_o^2 \sum_{r=1}^{n} m_r^2 \frac{e^{-p_{rr}}}{2} \int_0^\infty e^{-p_{rr}t}(1+t)^{1/2} t^{-1/2}\, dt$$
$$+ 2 \sum_{s=r+1}^{n} \sum_{r=1}^{n-1} m_r m_s \frac{e^{-p_{rs}}}{2} \int_0^\infty e^{-p_{rs}t}(1+t)^{1/2} t^{-1/2} \cos\left[q_{rs}(1+t)^{1/2}\right] dt, \tag{1}$$

where $p_{rr} = 2K_o f_r$; $p_{rs} = K_o(f_r+f_s)$; $q_{rs} = K_o(h_r-h_s)$; and h_r and f_r are the longitudinal and vertical position respectively of the r^{th} source. The parameter $K_o = g/c^2$, where c is the speed of the ship and g is the acceleration of gravity. The parameter ρ is the density of water, and the quantity m_r is the strength of the r^{th} source.

If we designate the first integral as $f(p)$ and the second as $g(p,q,0)$, then we may write

$$\frac{\partial R}{\partial m_r} = 16\pi\rho K_o^2 \left\{ m_r e^{-p_{rr}} f(p_{rr}) + \sum_{s\neq r} m_s e^{-p_{rs}} g(p_{rs},q_{rs},0) \right\} \tag{2}$$

We now have an expression from which the change in wave-making resistance can be calsulated as each of the m_r is allowed to vary.

A short table of the functions $f(p)$ and $g(p,q,0)$ has been calculated by the David Taylor Model Basin. In the problem as laid out here the values of p which will be required are $p = 0.1, 0.2, \ldots, 0.9$, since $K_oL = 10$ and so $.005K_oL = .05$, $.015K_oL = .15$, etc. The values of q which will be required are $0,1,2,\ldots,10$. Hence we will need some 99 tabular values, since $g(p,0,0) = f(p)$.

The rotation of vectors which will be carried through will consist simply of taking linear combinations of the $\frac{\partial R}{\partial m_r}$. To keep the first moment of the source strength constant and the uppermost level of sources unaffected, the following type of vector may be added to the source strengths:

0	0	0	0	0	0	0	0	0	0	0
-1/40	0	0	1/4	0	0	0	-1/4	0	0	1/40
-1/40	0	0	0	0	0	0	0	0	0	1/40
-1/40	0	0	0	0	0	0	0	0	0	1/40
-1/40	0	0	0	0	0	0	0	0	0	1/40

Table 2. Sample Vector Which Produces Unit Local Change in Longitudinal Moment of Source Strength.

There are 20 possible linearly independent vectors with fore-and-aft symmetry of the kind shown in the example above, and they will all be used. The sum of the source strengths added by any multiple of one of these vectors is zero, so the closure of the streamlines will not be affected.

The scale of the vectors which are made up of linear combinations of other vectors is arbitrary. Accordingly, it is necessary to choose some measure of size and adjust all of them to this measure. The local change in volume of the hull will be chosen as the measure of size in this calculation. The local moment change, which is proportional to the local volume change, in the sample vector is $(-1/4) \times 4 = -1$ if that vector is added to the sources defining the hull. The other vectors will be chosen to produce the same result -- i.e. instead of $+1/4$ and $-1/4$ separated by 4 spaces we might have $+1/6$ and $-1/6$ separated by 6 spaces, and so on. Such a vector would be like this:

0	0	0	0	0	0	0	0	0	0	0
-1/40	0	1/6	0	0	0	0	0	-1/6	0	1/40
-1/40	0	0	0	0	0	0	0	0	0	1/40
-1/40	0	0	0	0	0	0	0	0	0	1/40
-1/40	0	0	0	0	0	0	0	0	0	1/40

Table 3. Another Sample Vector.

The next task is to calculate the initial set of values of $\frac{\partial R}{\partial m_r}$ from Equation (2). Since only the relationship between the values is important, the coefficients ahead of the braces in Equation (2) may be omitted from the calculation. The coefficients $e^{-p_{rr}} f(p_{rr})$ and $e^{-p_{rs}} g(p_{rs}, q_{rs}, 0)$ will now be tabulated to simplify the work. We will let $w(p,q) = e^{-p} g(p,q,0)$. Since $f(p) = g(p,0,0)$, one table will serve the purpose.

q →

p	0	1	2	3	4	5	6	7	8	9	10
.1	10.95	.83	-3.37	-1.88	-.51	1.60	1.29	.49	-1.09	-1.07	-.49
.2	5.56	.73	-2.64	-1.64	-.46	1.37	1.15	.44	-0.95	-0.95	-.44
.3	3.66	.64	-2.09	-1.43	-.41	1.18	1.02	.39	-0.83	-0.85	-.40
.4	2.67	.57	-1.67	-1.25	-.37	1.02	0.91	.35	-0.73	-0.76	-.36
.5	2.06	.50	-1.35	-1.10	-.33	0.88	0.81	.32	-0.64	-0.68	-.32
.6	1.64	.44	-1.10	-0.96	-.30	0.76	0.72	.29	-0.56	-0.61	-.29
.7	1.34	.39	-.90	-0.85	-.27	0.65	0.64	.26	-0.49	-0.54	-.26
.8	1.11	.34	-.75	-0.74	-.24	0.57	0.57	.23	-0.43	-0.49	-.24
.9	0.93	.30	-.62	-0.66	-.22	0.49	0.51	.21	-0.38	-0.44	-.21

Table 4. Values of $w(p,q)$

In order to simplify the notation, the positions of the sources will be designated by values to the same scale as the variables p and q. This will give us a table of locations which looks like this:

.05,0	.05,1	.05,2	.	.	.	.	.	.05,9	.05,10
.15,0	.15,1	.15,2	.	.	.	.	.	.15,9	.15,10
.	.	.	.	.	.	.	.	.	.
.45,0	.45,1	.45,2	.	.	.	.	.	.45,9	.45,10

Table 5. Table of Locations of Sources in Dimensions of p and q.

The most work in the calculation is in obtaining the quantities $\frac{\partial R}{\partial m_r}$ at each step. Fortunately, it is possible to use the fore-and-aft symmetry of the source distribution together with the fact that the top row of sources are to be held constant in strength to reduce the number required from 55 down to 20 at each step. As an example the calculation for $\frac{\partial R}{\partial m_{.15,2}}$ is presented, where the subscript is in the notation used above.

$$\begin{aligned}\frac{\partial R}{\partial m_{.15,2}} &= m(.05,0)\, w(.2,2) + m(.05,1)\, w(.2,1) + m(.05,2)\, w(.2,0) \\ &\quad + m(.05,3)\, w(.2,1) + m(.05,4)\, w(.2,2) + m(.05,5)\, w(.2,3) \\ &\quad + m(.05,6)\, w(.2,4) + m(.05,7)\, w(.2,5) + m(.05,8)\, w(.2,6) \\ &\quad + m(.05,9)\, w(.2,7) + m(.05,10)\, w(.2,8) + \ldots\end{aligned}$$

$$= m(.05,0)\ [w(.2,2) - w(.2,8)] + m(.05,1)\ [w(.2,1) - w(.2,7)]$$
$$+ m(.05,2)\ [w(.2,0) - w(.2,6)] + m(.05,3)\ [w(.2,1) - w(.2,5)]$$
$$+ m(.05,4)\ [w(.2,2) - w(.2,4)] + \dots , \qquad (3)$$

where the remaining terms involve other values of p in the table for $w(p,q)$. Specifically we have the relation $m(.05,X) \to p = .2$; $m(.15,X) \to p = .3$; $m(.25,X) \to p = .4$; $m(.35,X) \to p = .5$; and $m(.45,X) \to p = .6$. This applies, of course, only for $\frac{\partial R}{\partial m_{.15,2}}$. Here X, Y vary from 0 to 10. Similar simplifications may be worked out for all the other 19 $\frac{\partial R}{\partial m_r}$ used in the calculation.

Derivative With Respect to the Basis Vectors

Once the derivative of R with respect to each of the m_r has been calculated, it is necessary to calculate the change in R which results from unit change in each of the vectors like that shown in Table 2. We may designate the vectors with script $\mathcal{M}_i$ to distinguish them from the sources. Then the required derivative is $\frac{\partial R}{\partial \mathcal{M}_i}$, where i has 20 values.

To identify the vectors we will describe them with the same coordinates as are used for the sources, taking the location of the isolated source in the vector (i.e. 1/4 in Table 3) as its identifying feature. Then we can obtain a quantity proportional to $\frac{\partial R}{\partial \mathcal{M}_i}$ by using the equation

$$\frac{\partial R}{\partial \mathcal{M}_i} \approx -\frac{1}{40}\left(\frac{\partial R}{\partial m_{.15,0}} + \frac{\partial R}{\partial m_{.25,0}} + \frac{\partial R}{\partial m_{.35,0}} + \frac{\partial R}{\partial m_{.45,0}}\right) + k_i \frac{\partial R}{\partial m_i} , \qquad (4)$$

where k_i is the multiplier in the vector for the "naming" source. In the vector in Table 3, the multiplier k_i is 1/4.

The qunatity $\frac{\partial R}{\partial \mathcal{M}_i}$ is calculated from Equation (4). In addition its square is tabulated and the sum of the squares, summed over all i , is compared with the corresponding quantity for the previous calculation. A decrease in this sum corresponds to an imporved form.

Having calculated each of the twenty $\frac{\partial R}{\partial \mathcal{M}_i}$, we next multiply it by the quantity Δt , which is chosen so that it will produce a decrease in the resistance if each of the vectors is multiplied by $\frac{\partial R}{\partial \mathcal{M}_i} \Delta t$ and

added to the source strengths of the previous iteration. To do this we adied the sum $-\frac{1}{40}\sum_i \frac{\partial R}{\partial \eta_i}\Delta t$ to each of the $m_{.15,0}$, $m_{.25,0}$, $m_{.35,0}$ and $m_{.45,0}$. In addition we add the quantity $k_i \frac{\partial R}{\partial \eta_i}\Delta t$ to the i^{th} source. This gives us the source strength for the next ineration.

RESULTS OF THE CALCULATION

The first step in the calculation was to find the derivatives $\frac{\partial R}{\partial m_i}$, which show the change in resistance with unit change in each of the sources whose strength is permitted to vary.

Depth (Units of p)	Longitudinal Position 0	1	2	3	4
.15	5.50	.23	-.49	-.51	-.26
.25	3.94	.20	-.44	-.52	-.27
.35	3.04	.18	-.37	-.50	-.26
.45	2.44	.16	-.30	-.46	-.24

Table 6. Initial Values of $\frac{\partial R}{\partial m_i}$.

This table tells us immediately that a small increase in the strength of any of the sources in columns 2, 3, or 4 will provide a small decrease in the wave-making resistance of the ship. Such information may be useful in itself.

From Table 6 it was a simple matter to calculate the quantities $\frac{\partial R}{\partial \eta_i}$ using Equation (4). The initial values of these partial derivatives are shown in Table 7.

Depth (Units of p)	Longitudinal Position of "naming source" 0	1	2	3	4
.15	.27	-.27	-.89	-1.20	-1.51
.25	.03	-.32	-.86	-1.22	-1.54
.35	-.10	-.34	-.81	-1.18	-1.50
.45	-.20	-.37	-.76	-1.14	-1.43

Table 7. Initial Values of $\frac{\partial R}{\partial \eta_i}$.

$$\sqrt{\sum_i \left(\frac{\partial R}{\partial \eta_i}\right)^2} = 4.24$$

To find an improved hull, the entries of Table 7 were each multiplied by the quantity Δt , which was chosen to be -0.2. Then this product was multiplied into each of the terms of the vector with the same naming coordinate, and the result was added to the initial source strength given in Table I. A new set of source strengths was the result. From this set the process was repeated once more and the result was the following set of source strengths.

Depth (Units of p)	Longitudinal Position → 0	1	2	3	4	5	6	7	8	9	10
.05	.66	.10	.25	.20	.14	0	-.14	-.20	-.25	-.10	-.66
.15	.54	.12	.31	.28	.28	0	-.28	-.28	-.31	-.12	-.54
.25	.54	.12	.30	.29	.31	0	-.31	-.29	-.30	-.12	-.54
.35	.55	.12	.30	.29	.32	0	-.32	-.29	-.30	-.12	-.55
.45	.55	.12	.30	.29	.32	0	-.32	-.29	-.30	-.12	-.55

Table 8. Source Strengths of Improved Hull.

As might have been expected from an inspection of the first set of partial derivatives, the effect of following the gradient was to move volume into the midships section. The top row of sources, of course, did not change because the calculation had been set up to prevent this from happening.

The quantity $\sqrt{\sum_i \left(\frac{\partial R}{\partial m_i}\right)^2}$ is proportional to the gradient of the wave-making resistance in the vector space in which the calculation is carried out. Therefore the smaller this quantity the closer to a stationary value is the calculation. For the improved hull the value was 2.16, compared with an initial value of 4.24. Wave-making resistance was 0.66 times that of the original hull.

It can be shown that the closer to a stationary value the calculation gets the larger is the change in hull shape needed to provide a given decrease in resistance. Hence less result could be expected for the same amount of change. Two more steps, just like the first two, were taken and produced the set of source strengths shown below.

	Longitudinal Position										
Depth	0	1	2	3	4	5	6	7	8	9	10
.05	.66	.10	.25	.20	.14	0	-.14	-.20	-.25	-.10	-.66
.15	.47	.13	.35	.31	.19	0	-.19	-.31	-.35	-.13	-.47
.25	.49	.14	.35	.33	.29	0	-.29	-.33	-.35	-.14	-.49
.35	.50	.14	.34	.34	.34	0	-.34	-.34	-.34	-.14	-.50
.45	.50	.14	.33	.34	.37	0	-.37	-.34	-.33	-.14	-.50

Table 9. Source Strengths of Second Improved Hull.

For this second improved hull $\sqrt{\sum_i \left(\frac{\partial R}{\partial m_i}\right)^2} = 1.64$, compared with 4.24 for the original set of sources and 2.16 for the first improved hull. The resistance for this hull was calculated using Equation (1) and compared with that of the original set of sources. It turned out to be 0.55 times that of the original set -- that is, a decrease of 45 percent.

This set of sources does not represent a hull of minimum wave-making resistance, merely one whose resistance is considerably reduced from that of the hull represented by the initial set of sources. It has already started to develop an unusual feature -- a wide bulge at the midships section at the lowest waterline -- and in the next iteration this will clearly become more pronounced. At this point the naval architect might well conclude that further decrease in wave-making resistance is not worth the decrease in ability to come alongside a pier with such a bulge.

CONCLUSIONS

From this simple example, it is apparent that the method of steep descent can be used to decrease the wave-making resistance of a given hull. It seems reasonable to conclude that more could be accomplished if a digital computer rather than a slide rule were the calculating machine. The most important feature of the method appears to be its ability to find improvements which work some decrease in wave-making resistance without at the same time forcing the acceptance of hull shapes which are unreasonable based on other design criteria.

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AN ANALYSIS OF WAVE PROFILE ALONGSIDE THE SHIP

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INTRODUCTION

The purpose of this study is to calculate the wave profile alongside a ship represented by a singularity distribution over its centerline plane, in particular, a distribution of draftwise uniform strength. The above mentioned type of distribution for representing a ship has been the object of various complaints not only from naval architects but also from theorists. The most serious point is that the hull form obtained by tracing stream lines usually yields a rocker keel in contrast to the flat keel of a conventional ship hull; a poor representation of ship hull within such a restricted form as the above singularity distribution. In this connection, an important contribution has been made by Hess and Smith who have calculated the flow about a series 60 hull having a block coefficient of 0.60.(1) A paper connected to their work is to be presented by J. P. Breslin and King Eng.

In spite of its defects, the singularity distribution of draftwise uniform strength retains much that is desirable for the basic investigation of ship hydrodynamics as well as the refinement of practical ship hulls. This particular form of distribution is much easier to work with than is a more general form of distribution over curved surfaces, like the one Hess and Smith have worked with, and yet retains the ship surface boundary conditions which do not depend upon the so called thin-ship approximations. This fact is desirable in checking the validity of linearized boundary conditions which the wave theory has employed and in clarifying the effects produced by real fluids. The shortcomings and the corrections of theoretical studies on ship waves should be clarified and performed based on wave observations or the wave-analysis, as we call it, rather than trying modifications directly for coincidence of wave-making resistance between calculations and test results. A coherent modification of the theories will be obtained only through wave-analysis. This is due to the fact that ship waves are the direct physical phenomena following an advancing ship and suffer far less from the ambiguous effects of viscosity of the media than does wave-making resistance. The distribution of a draftwise uniform strength is also simple enough to provide us with an intuition into the distinctive features of the wave-making characteristics of the correlated ship hull. Other forms of distribution, despite better quantitative prediction of performance, are unable to do this, and

therefore are of little help to naval architects. Moreover, a careful study even within the restriction of the simplest distribution of draftwise uniform strength has succeeded in a remarkable reduction of wave-making resistance.(2)

Influenced by these considerations, the author has engaged in calculating wave profiles alongside models and wave patterns following advancing models. The cooperative work pertaining to the calculation of wave profiles had been undertaken by the Wave-Making Resistance Subcommittee of the Japan Towing Tank Committee in 1960, when the author was in Japan and participated in it. Since coming here to The University of Michigan a year later, the author has continued his endeavors in the same direction by utilizing the facilities of the University.

A discussion of this paper pertaining primarily to wave profile calculations for various patterns of the singularity distribution and comparisons between evaluated wave profiles and measured ones. An example of the approach to ship hydrodynamics using the singularity distribution of draftwise uniform strength is also to be presented in this seminar by T. Inui, titled "Non-Bulbous Hull Forms Derived from Source Distribution on the Vertical Central Rectangular Plane."

WAVE PROFILE CALCULATION

The models with which this paper is concerned can be represented by a singularity distribution over a vertical central rectangular plane, extending to a certain finite depth with uniform strength. The program to evaluate the surface elevation alongside the above mentioned models is made in terms of the MAD language (Michigan Algorithem Decoder) so as to be able to figure out wave profiles at desired speeds based on singularity distribution, which are the only necessary data.

The right-hand Cartesian coordinate system (ξ, η, ζ) is fixed in the model with the origin at the midship.

The source distribution function is in the form of a product of two functions each dependent upon only one of the two space coordinates of the vertical center plane.

$$
\begin{aligned}
&m(\xi', \zeta') = m_1(\xi') \cdot m_2(\zeta') \cdot U \\
&m_1(\xi') && -\ell \leq \xi' \leq \ell \quad (\ell = L/2) \\
&m_2(\zeta') = 1 && -T \leq \zeta' \leq 0 \qquad (1)
\end{aligned}
$$

where U is the undisturbed uniform velocity of the model advancing in the positive direction of ξ-axis, L is model length and T is the depth of the singularity distribution.

Putting

$$x = Ko\xi, \qquad x' = Ko\xi', \qquad t = -Ko\zeta'$$

$$x_o = KoL, \qquad t_o = KoT$$

then the wave profile on the model central plane, $\zeta(x)$, is given as

$$\zeta(x) = \lim_{\mu \to +0} \frac{1}{K_o^2 \pi i} \int_0^{t_o} dt \int_{-\frac{x_o}{2}}^{x_o/2} m_1(x')dx' \int_{-\pi}^{\pi} \int_0^{\infty} e^{-Kt+iK(x-x')\cos\Theta} \frac{K \cos\Theta \, dK d\Theta}{K\cos^2\Theta - 1 + i\mu\cos\Theta} d\Theta \qquad (2)$$

where $K = Ko \sec^2\Theta$.

The calculated wave profile on the model central plane may be substituted for the wave profile alongside the model surface because of the stationary property of the wave pattern in the vicinity of the model center line.

$\zeta(x)$ may be expressed in terms of the contribution due to the element of the distributed singularities on the plane. By using the functions described in the Appendix,

$$\begin{aligned}\zeta(x) &= \frac{4}{K_o^2} \int_0^{t_o} dt \int_{-\frac{x_o}{2}}^{x_o/2} m_1(x')dx'[\frac{\partial}{\partial t} O_{-1}^{(1)}(x - x', t)] \\ &= \int_{-\frac{x_o}{2}}^{\frac{x_o}{2}} m_1(x') [O_{-1}^{(1)}(x-x',0) - O_{-1}^{(1)}(x-x',t_o)] \, dx'\end{aligned} \tag{3}$$

where $x \leq 0$, $O_{-1}^{(1)}(x,t)$ is a monotonic function, but for $x < 0$ it contains an oscillating term, $P_{-1}(x,t)$.

$$O_{-1}^{(1)}(x,t) = -[O_{-1}(-x,t) - 2P_{-1}(-x,t)] \qquad x < 0 \tag{4}$$

Explicit expressions of the wave profile are

(a) $x \geq \frac{x_o}{2}$; forward of the FP

$\zeta(x)$ is given as it is in (3)

(b) $\frac{x_o}{2} \geq x \geq -\frac{x_o}{2}$; along model center plane

$$\begin{aligned}\zeta(x)/L = 1/\pi x_o \Bigg[&\int_{-\frac{x_o}{2}}^{x} m_1(x') [O_{-1}^{(1)}(x-x',0) - O_{-1}^{(1)}(x-x',t_o)] \, dx' \\ &- \int_x^{\frac{x_o}{2}} m_1(x')[O_{-1}^{(1)}(x'-x,0) - O_{-1}^{(1)}(x'-x,t_o) \\ &- 2[2P_{-1}(x'-x,0) - 2P_{-1}(x'-x,t_o)]] \, dx' \Bigg]\end{aligned} \tag{5}$$

(c) $x \leq -\frac{x_o}{2}$; aft of the AP

$$\zeta(x)/L = 1/\pi x_o \left[\int_{-\frac{x_o}{2}}^{x_o/2} m_1(x') \, [O_{-1}^{(1)}(x'-x,0) - O_{-1}^{(1)}(x'-x,t_o) \right.$$

$$\left. - 2\,[P_{-1}(x'-x,0) - P_{-1}(x'-x,t_o)]]\, dx' \right] \qquad (6)$$

Among the terms involved in the above equations, the ones consisting of $O_{-1}^{(1)}$ functions pertain to a local disturbance component and the others consisting of P_{-1} functions to free travelling wave components. The former advances along with the model and has nothing to do with wave resistance. Only the latter contributes to wave-making resistance. In case of an infinite draft model the terms with the parameter of t_o vanish.

The current computer program is made so as to yield print outs of wave height at succeeding stations spaced 1/20 of model length and in terms of a local disturbance, a free travelling wave and a total wave. At each station numerical integration is carried out referring to the functions concerned. The contribution of the singularity in the vicinity of the station is significant, in particular, the functions $O_{-1}^{(1)}(x-x', 0)$ and $P_{-1}(x-x', 0)$ yield infinite logarithms when x equals to x'. Evaluation of an integral near such a logarithmic singularity is approximated as follows(3) and the identical treatment is applied to the other singularities.

$$\int_{x-\epsilon}^{x} m_1(x')O_{-1}^{(1)}(x-x',0)dx' = \int_{0}^{\epsilon} m_1(x-u)\left[\frac{O_{-1}^{(1)}(u,0)}{\log u}\right] \log u du$$

$$= -\frac{\epsilon}{6}\left[(\frac{17}{6}\log\epsilon)\, f_o + (\frac{20}{6} - 4\log\epsilon)\, f_1 - (\frac{1}{6} + \log\epsilon)\, f_2\right]$$

where

$$f_{o,1,2} = \left| m_1(x-u)\,\frac{O_{-1}^{(1)}(u,0)}{\log u}\right|_{u=0,\ \frac{\epsilon}{2},\ \epsilon}$$

and

$$\lim_{u\to +o} O_{-1}^{(1)}(u,o) = -\frac{1}{2}\log u$$

as for $P_{-1}(u,o)$,

$$\lim_{u\to +o} P_{-1}(u,o) = -\log u$$

ϵ is taken as $0.025x_o$ in the current program. For the numerical integration over the rest of the part, the Simpson's first rule is commonly used in accordance with the sections in the adjoining table.

Section	Range of (x-x') or (x'-x)	No. of Application of Simpson's Rule
The first next section to ϵ	$(0.025-0.050)x_o$	1
The second next section to ϵ	$(0.05-0.01)x_o$	1
The remaining section	Spacing of ordinate	$0.05x_o$

The functions, $O_{-1}^{(1)}(u,t)$ and $P_{-1}(u,t)$, involved in the numerical integration are calculated in their subroutines according to the parameters, u and t, and called up into the wave profile calculation program. Although numerical integration is commonly used in evaluation of the above mentioned functions, asymptotic expansions are made use of in accordance with smaller or larger values of the variable of u for a particular value of t. The actual procedure to evaluate the values of the functions will be explained in another paper, "Some Mathematical Tables for the Determination of Wave Profiles," of this seminar. The method used herein is essentially the same. The accuracy of the values of the functions used in wave profile calculations to at least four or more significant figures is attained. Accordingly, the accuracy of the obtained wave profiles remains at about three significant figures, which is considered comparable with the accuracy of measured wave profiles.

WAVE PROFILES GENERATED BY VARIOUS SINGULARITY DISTRIBUTIONS

The wave-making characteristics of a ship are closely related to a geometrical configuration of singularity distribution representing the ship hull. Wave profiles were calculated with typical variations of singularity distributions. In making these variations of the distribution, the total flux out of the singularity distributed on the fore half plane and the depth of the distribution was kept constant, taking model length as two,

$$\int_0^1 m(\xi)\, d\xi = 0.44,$$

and the depth length ratio as 0.05. As far as functional expression of the distribution $m(\xi)$ was concerned, the fifth order polynomial was made use of. The hull configuration of moderate speed freighters, say a Series 60 with block coefficient of .70, and of modern tankers, were borne in mind in determining the singularity distributions used. Series F models in Table 1 are for the former and Series T models in Table 2 for the latter.

The geometry of the load water line was obtained for each singularity distribution by means of stream line tracing and, as shown in Figures 1 and 2, are compared with those of conventional ships. The frame line yielded a U-shape, which varied only slightly for each distribution, the bottom having a rocker keel. Hull configuration due to a singularity distribution of draftwise uniform strength has previously been investigated in detail.(4) As is shown in ship wave theory, the wave profile at low through moderate speeds is influenced mostly by geometry of the ship entrance or by the shape of the sectional area curve at the entrance, in other words by the ship form within a distance from the F. P. roughly equal to the length of the generated wave. Wave profiles of the above mentioned models are shown at two speeds of K_oL = 20 and 24 (speed length ratio = 0.753 and 0.685 respectively) in Figures 3 and 4 for Series F, and at a speed of K_oL = 30 (speed length ratio = 0.615) in Figure 6 for Series T. Each figure consists of a local disturbance and a total free travelling wave. The former is symmetrical with respect to midship, because each model is made up of a symmetrical source distribution.

Models F2 through F4 have fewer hollows and humps in the wave profile within the model length than does Model F1, and have the deepest trough about at the midship.

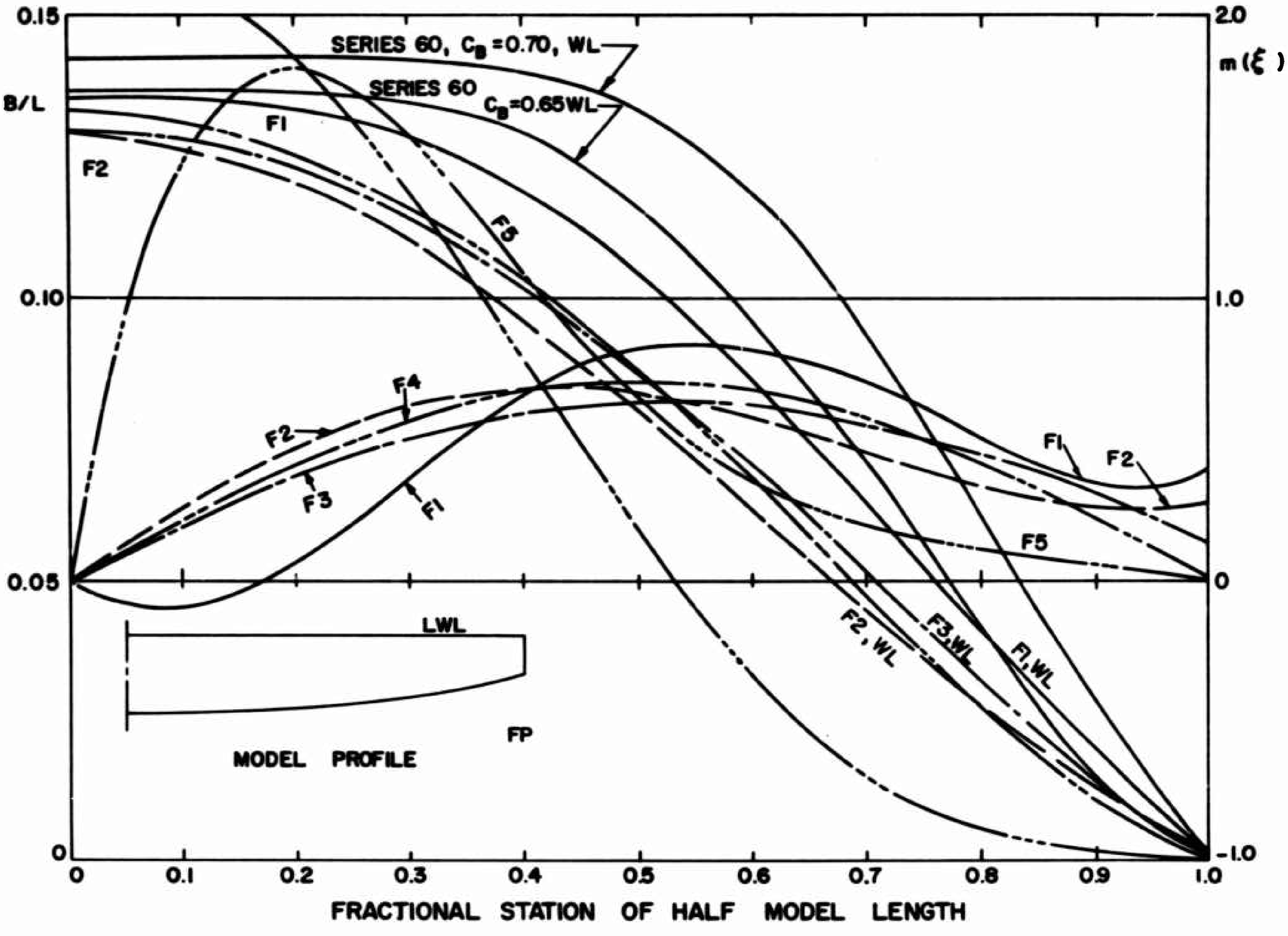


Figure 1. Singularity distributions and load water lines of series F models.

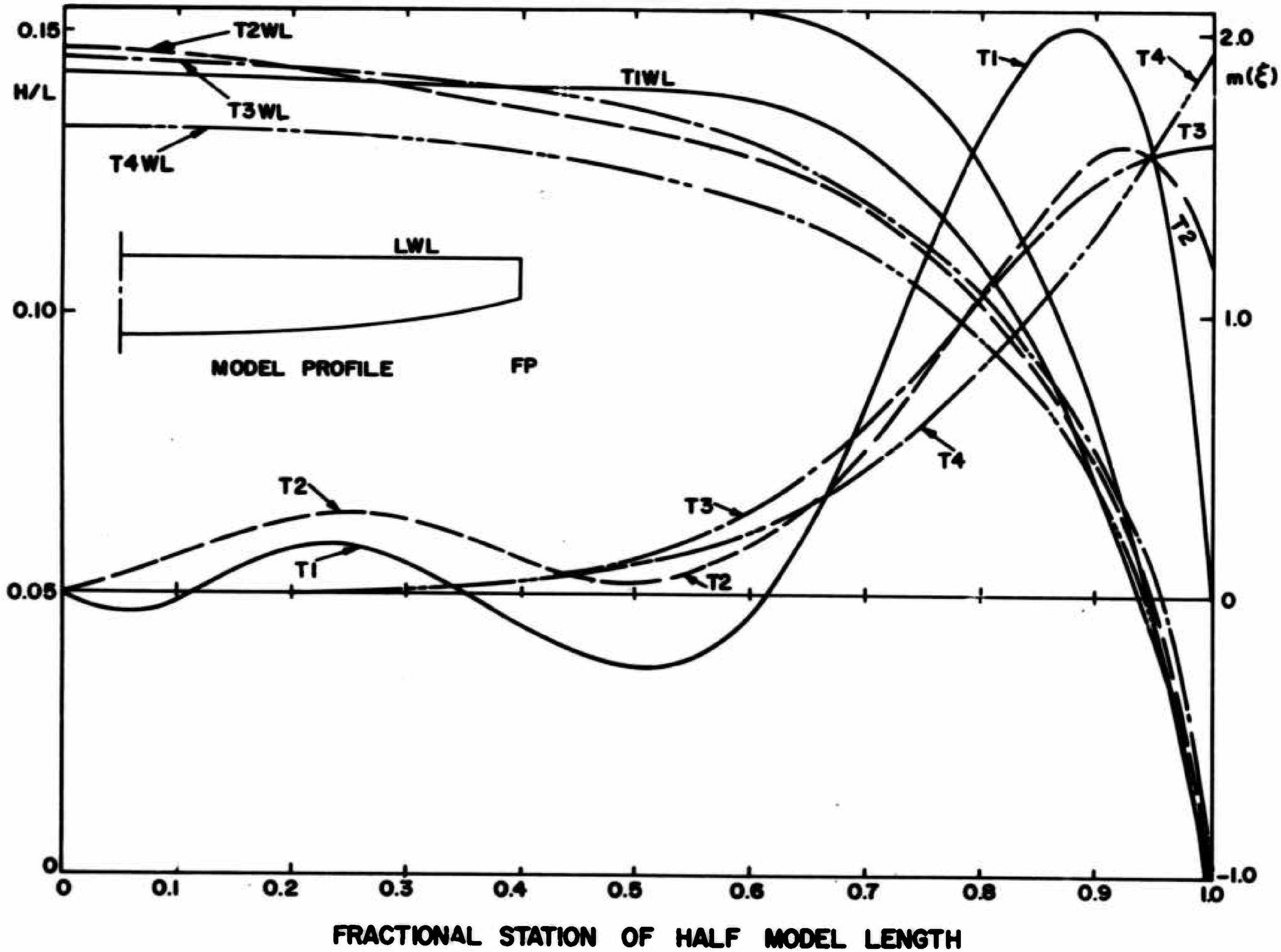
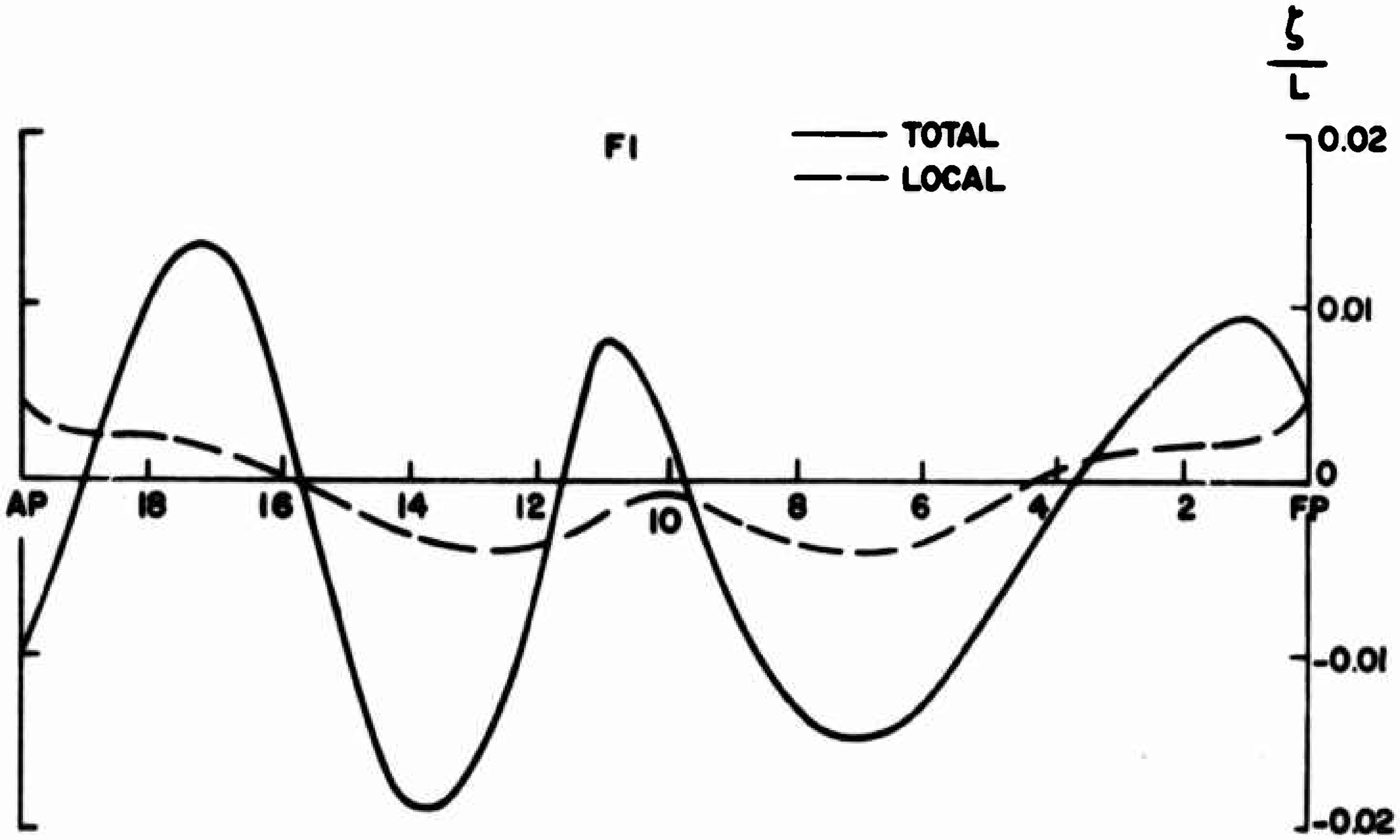


Figure 2. Singularity distributions and load water lines of series T models.



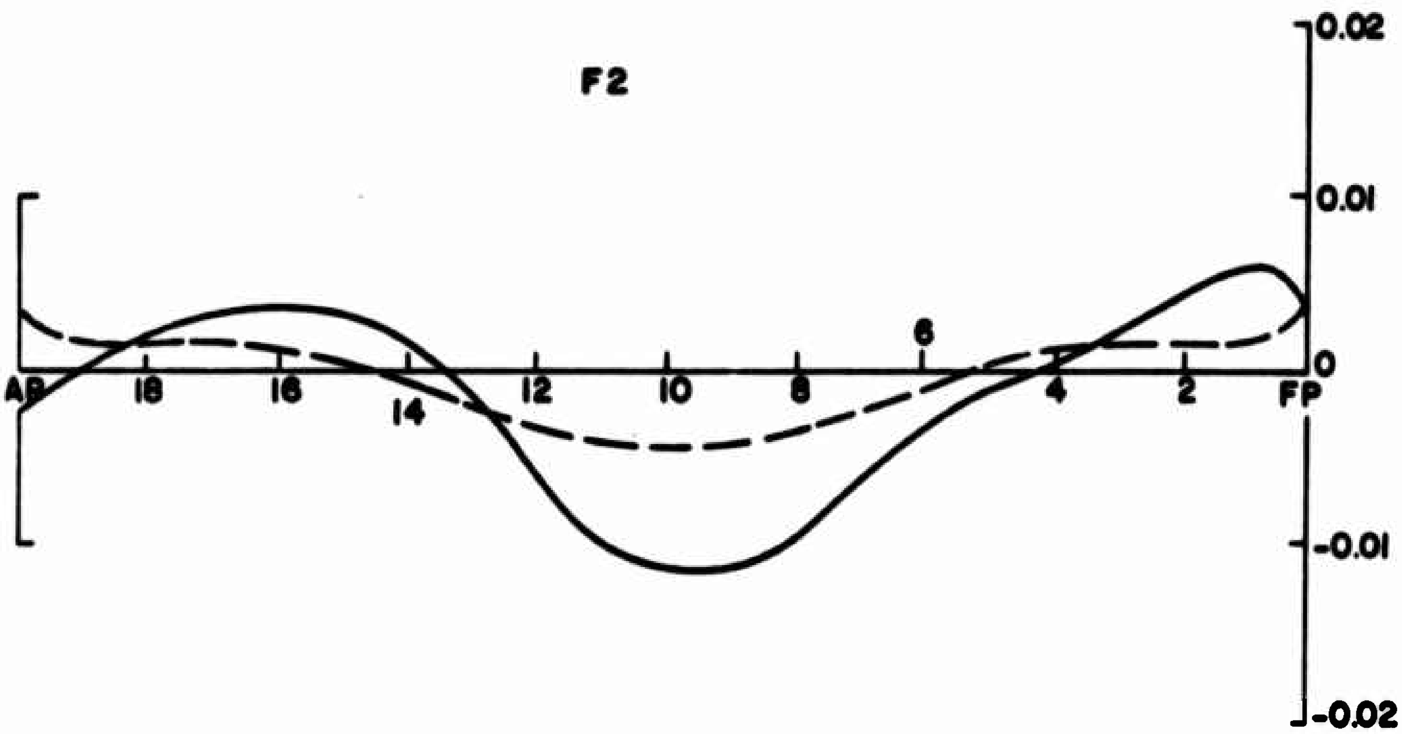


Figure 3. Calculated wave profiles of Series F models at a speed of KoL = 20 (speed length ratio = 0.753). (Continued).

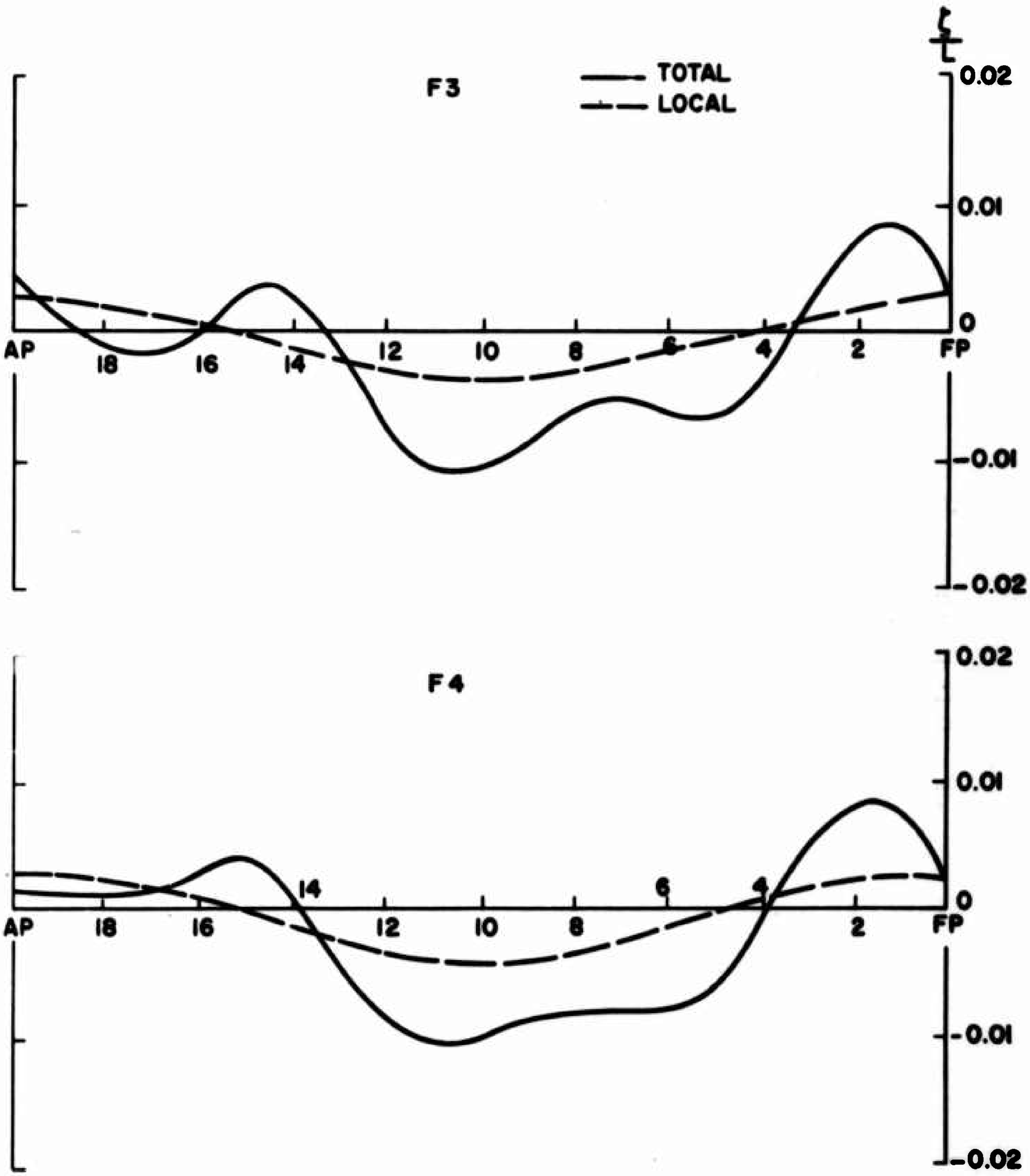
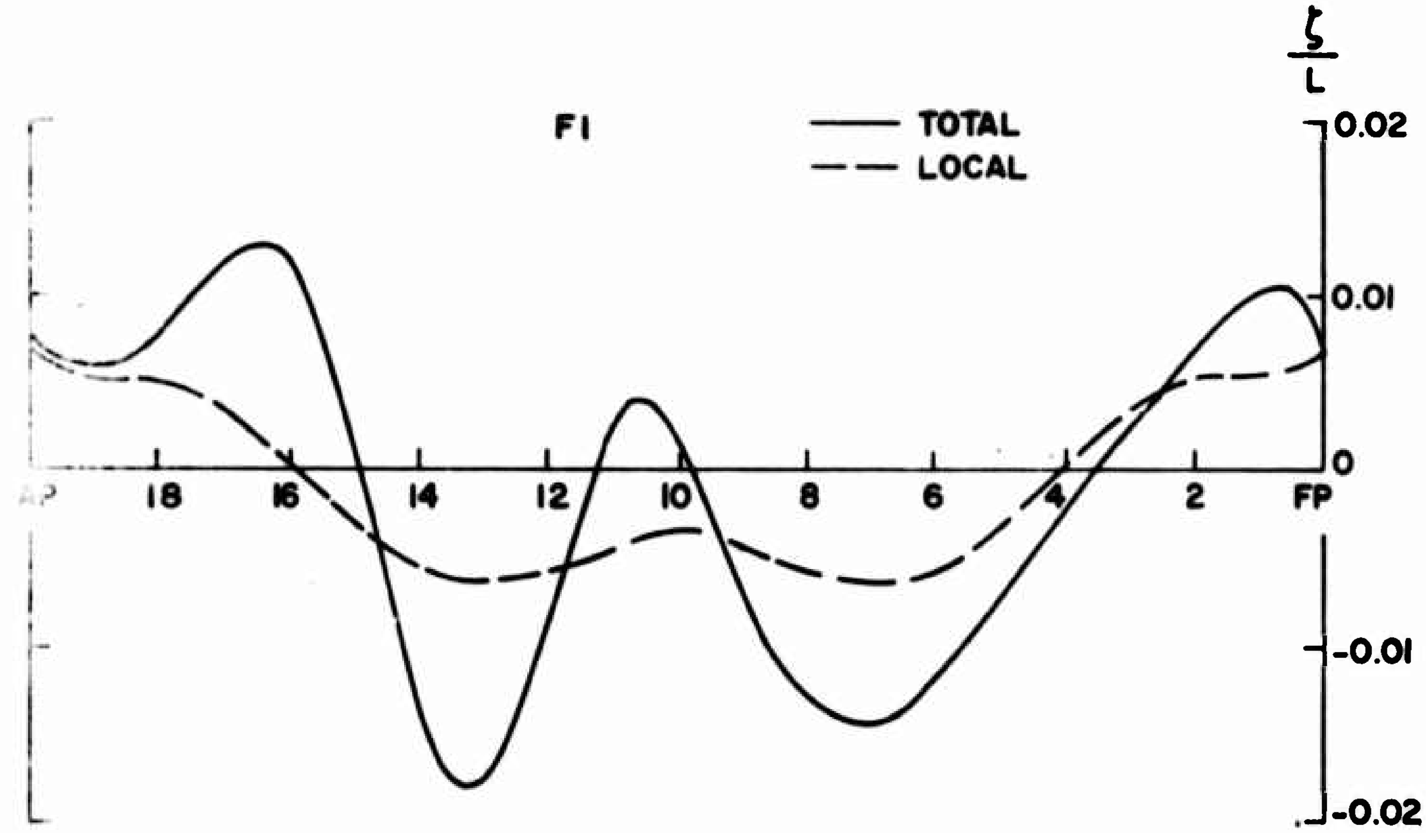


Figure 3 (cont'd.). Calculated wave profiles of Series F models at a speed of KoL = 20 (speed length ratio = 0.753).



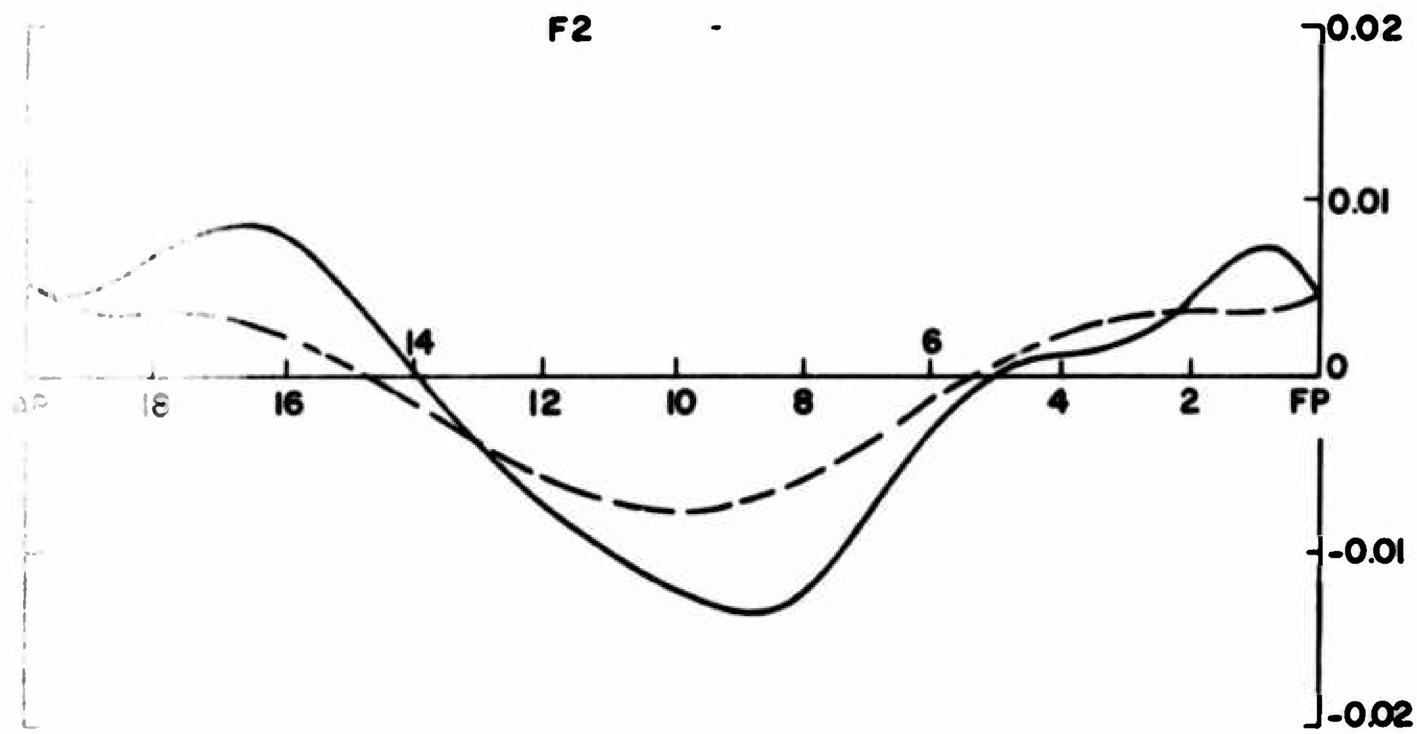
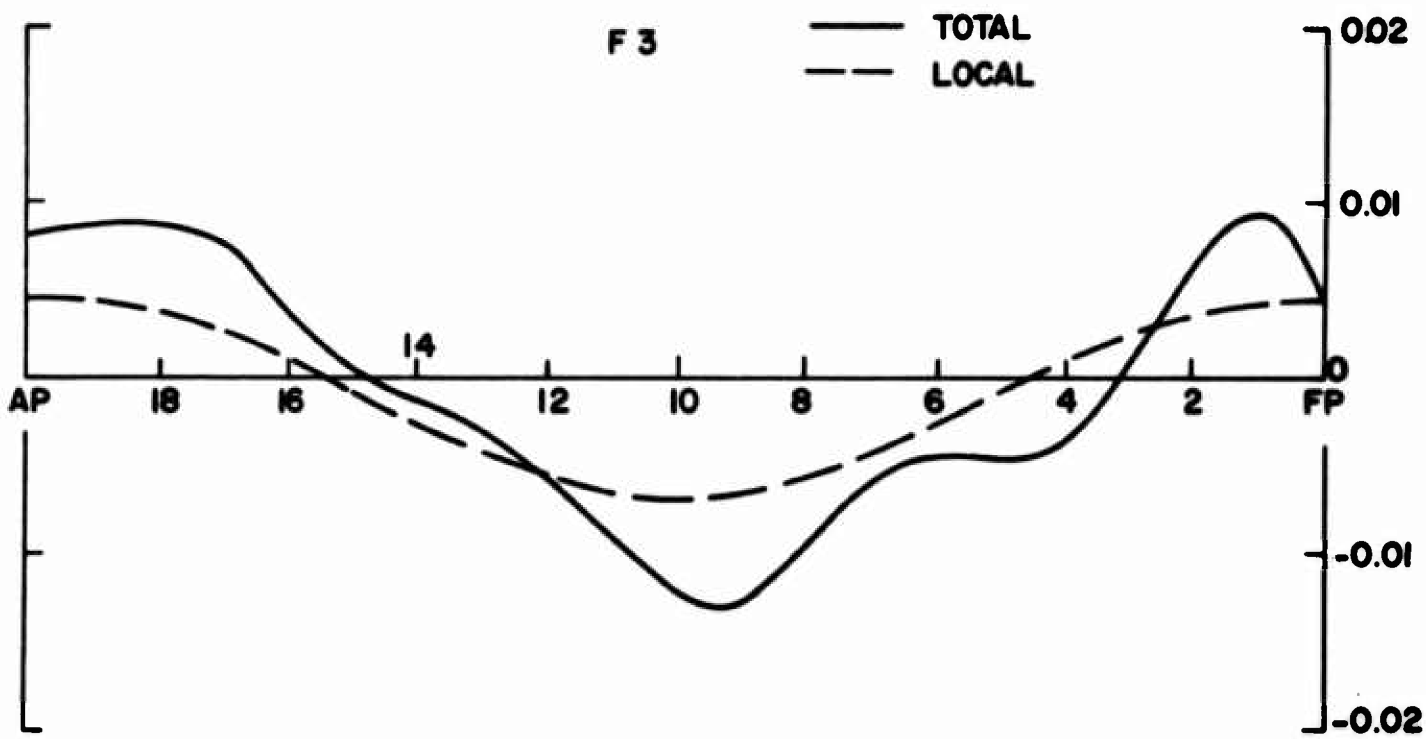


Figure 4. Calculated wave profiles of Series F models at a speed of KoL = 24 (speed length ratio = 0.685). (Continued).



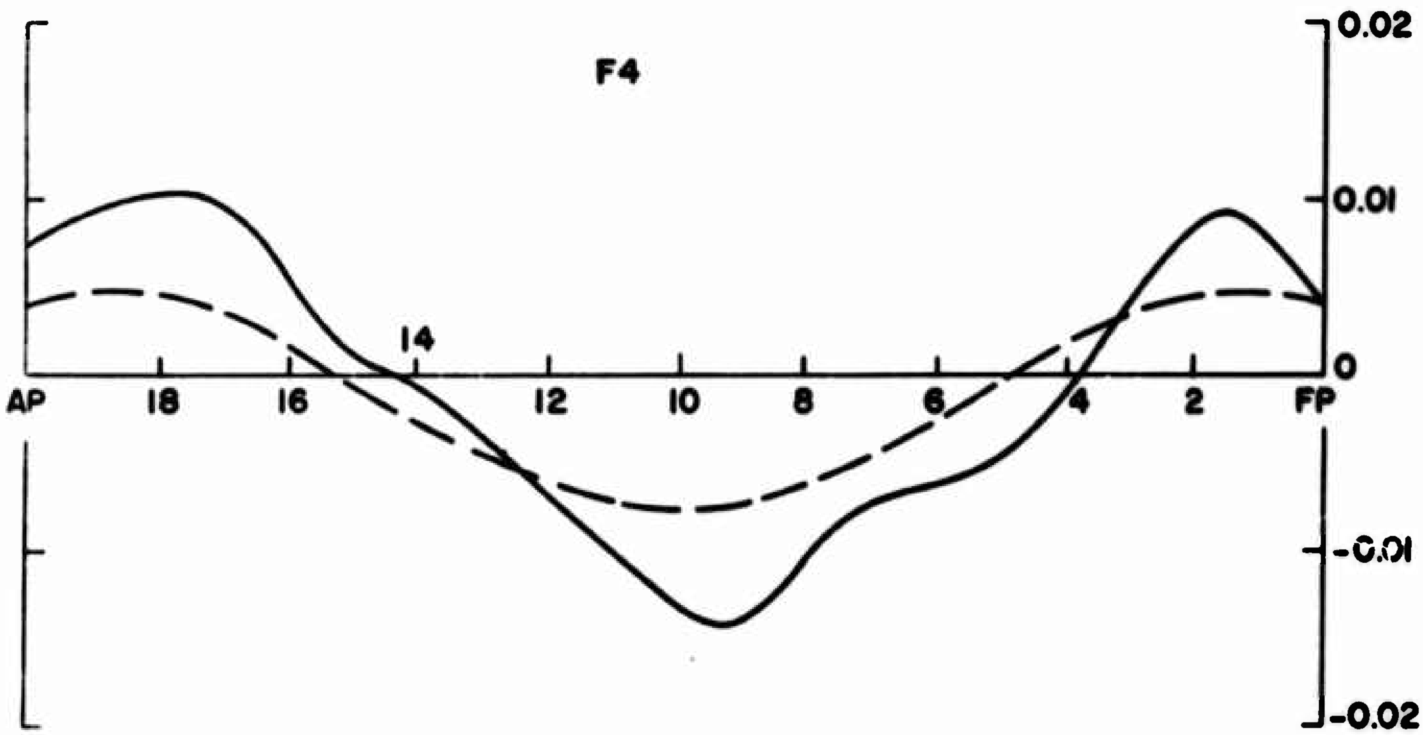


Figure 4 (cont'd). Calculated wave profiles of Series F models at a speed of KoL = 24 (speed length ratio = 0.685).

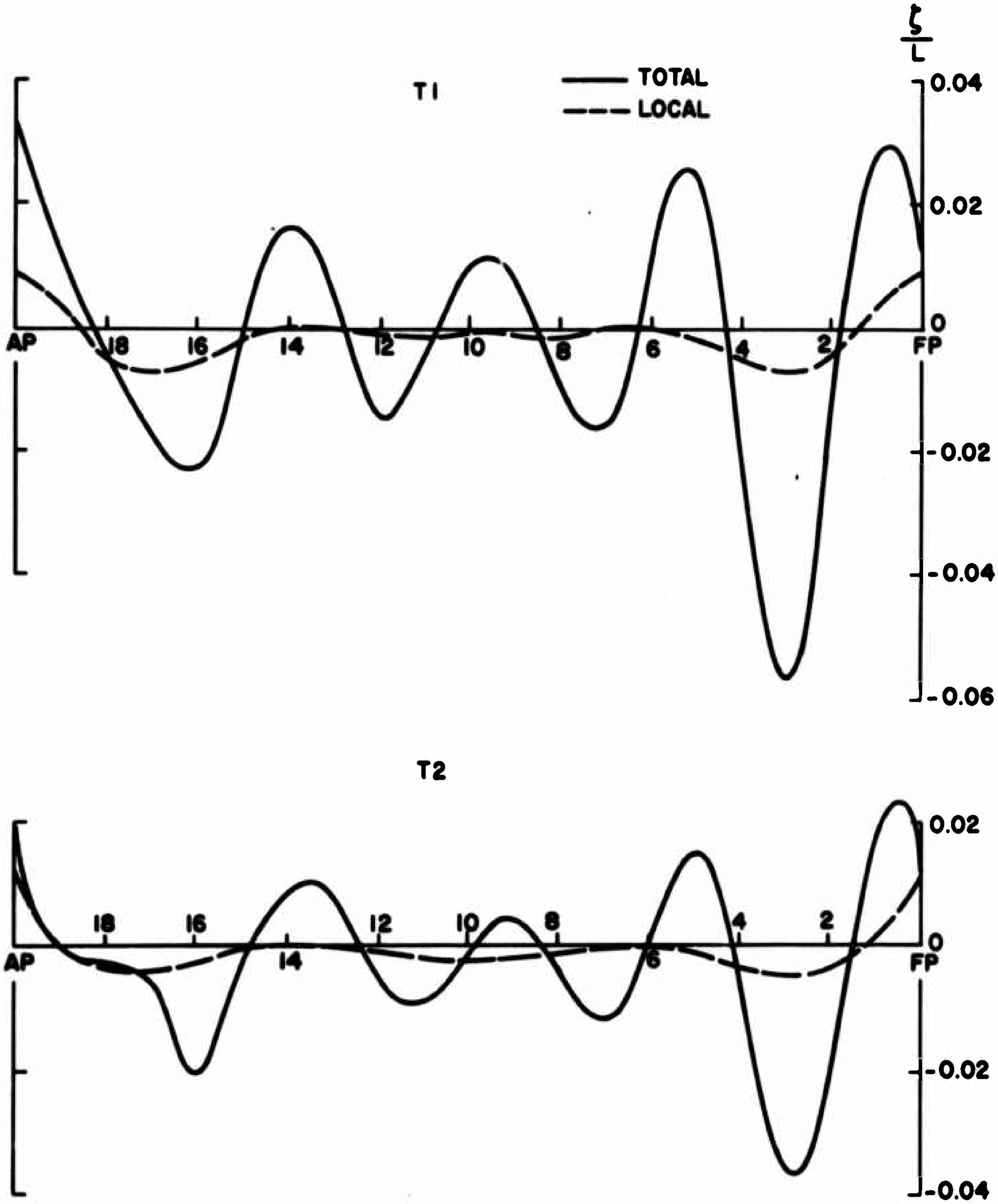
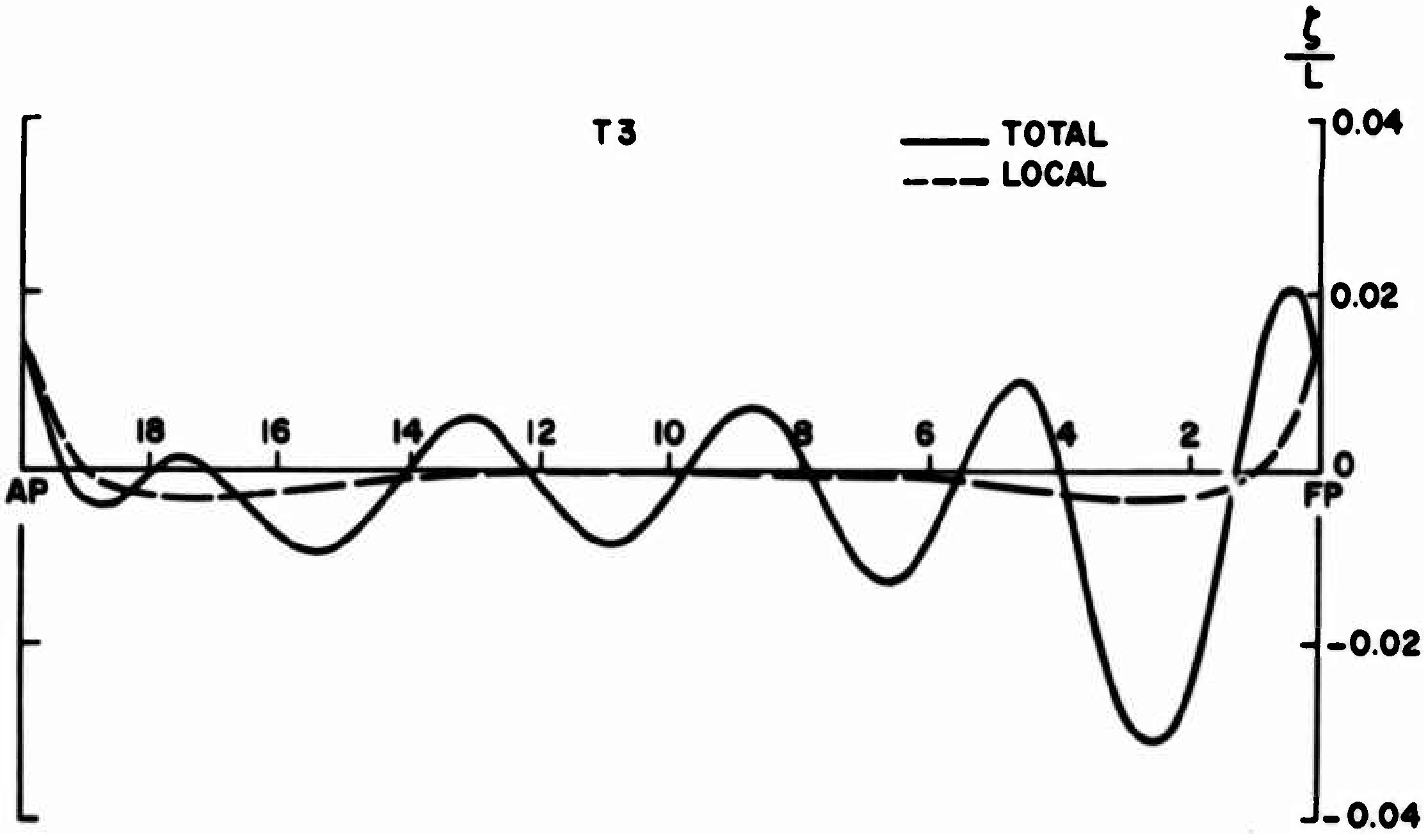


Figure 5. Wave profiles of Series T models at a speed of KoL = 30 (speed length ratio = 0.615). (Continued).



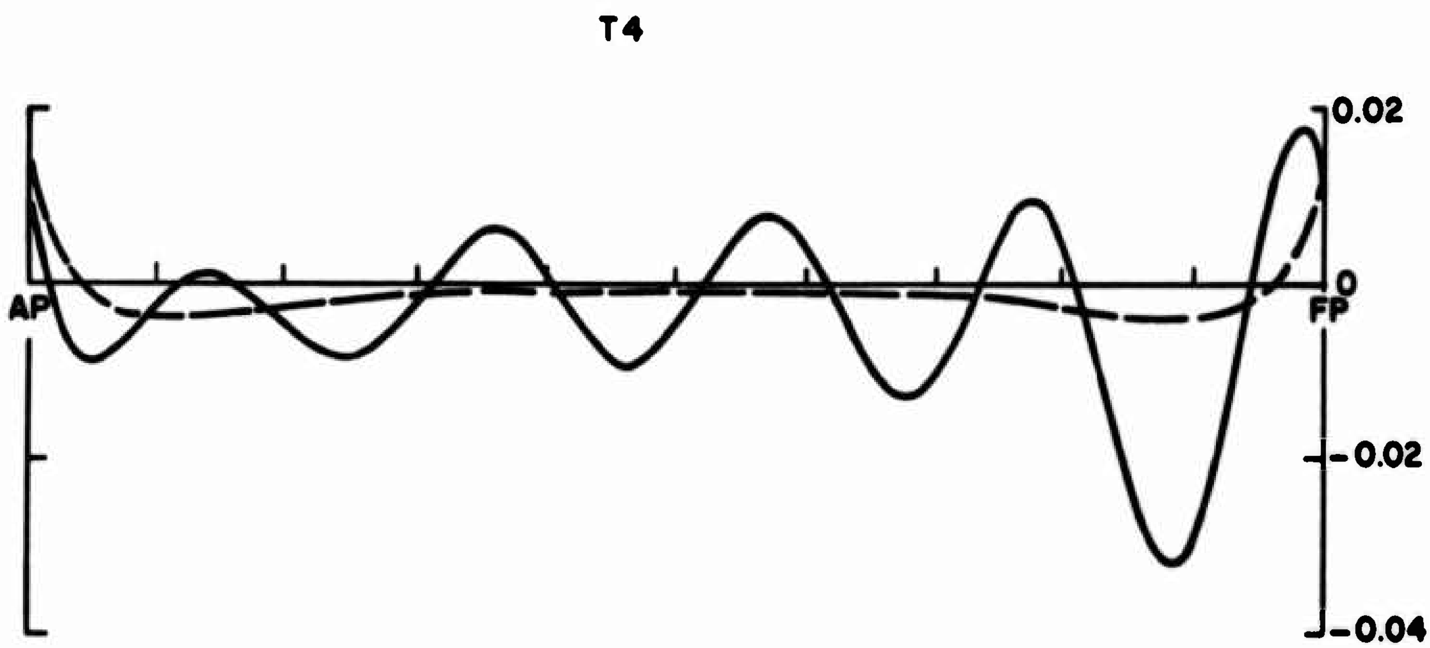


Figure 5 (cont'd.). Wave profiles of Series T models at a speed of KoL = 30 (speed length ratio = 0.615).

Series T models are more or less similar in their wave profiles. The wave height increases in accordance with the entrance angle as does the wave resistance. The relationship among these three quantities appears to be much simpler than with the Series F models.

The contribution to wave-making resistance due to the bow wave alone was evaluated for each model from the following formulas.

$$R_{WB} = \pi\rho U^2 \int_0^{\pi/2} [A_F(\theta)] \cos^3\theta d\theta$$

$$[A_F(\theta)]^2 = [A_{sin}(\theta)]^2 + [A_{cos}(\theta)]^2$$

Where $A = (\theta)$ is the amplitude function of the bow wave. In case $m(\xi)$ as well as its higher derivatives are continuous, its sine and cosine components A_F sin and A_F cos are given as follows.

$$A_F \sin/L = \frac{\sec\theta \cdot Z}{\pi}\left[\frac{m_o}{K_oL \sec\theta} - \frac{4m_2}{(K_oL \sec\theta)^3} + \frac{16m_4}{(K_oL \sec\theta)^5} \text{---}\right]$$

$$A_F \cos/L = \frac{\sec\theta \cdot Z}{\pi}\left[\frac{2m_1}{(K_oL \sec\theta)^2} - \frac{8m_2}{(K_oL \sec\theta)^4} + \frac{32m_5}{(K_oL \sec\theta)^6} \text{---}\right]$$

where $Z = 1 - e^{-K_oT \sec^2\theta}$

and

$$m_n = \left|\frac{d^n m(\xi)}{d\xi^n}\right|_{\xi=1}$$

The calculated resistance is given in terms of a coefficient

$$C_{WB} = R_{WB}/(\tfrac{1}{2}\rho U^2 L^2)$$

Figure 6 pertains to Series F and Figure 7 to Series T.

In order to obtain the total resistance of the model, the wave-making resistance of the stern wave alone and the interference effect between the bow wave and the stern wave are added. The former is identical with that of the bow wave considered independently composes the monotonic term of the resistance. The latter contains an oscillating term.

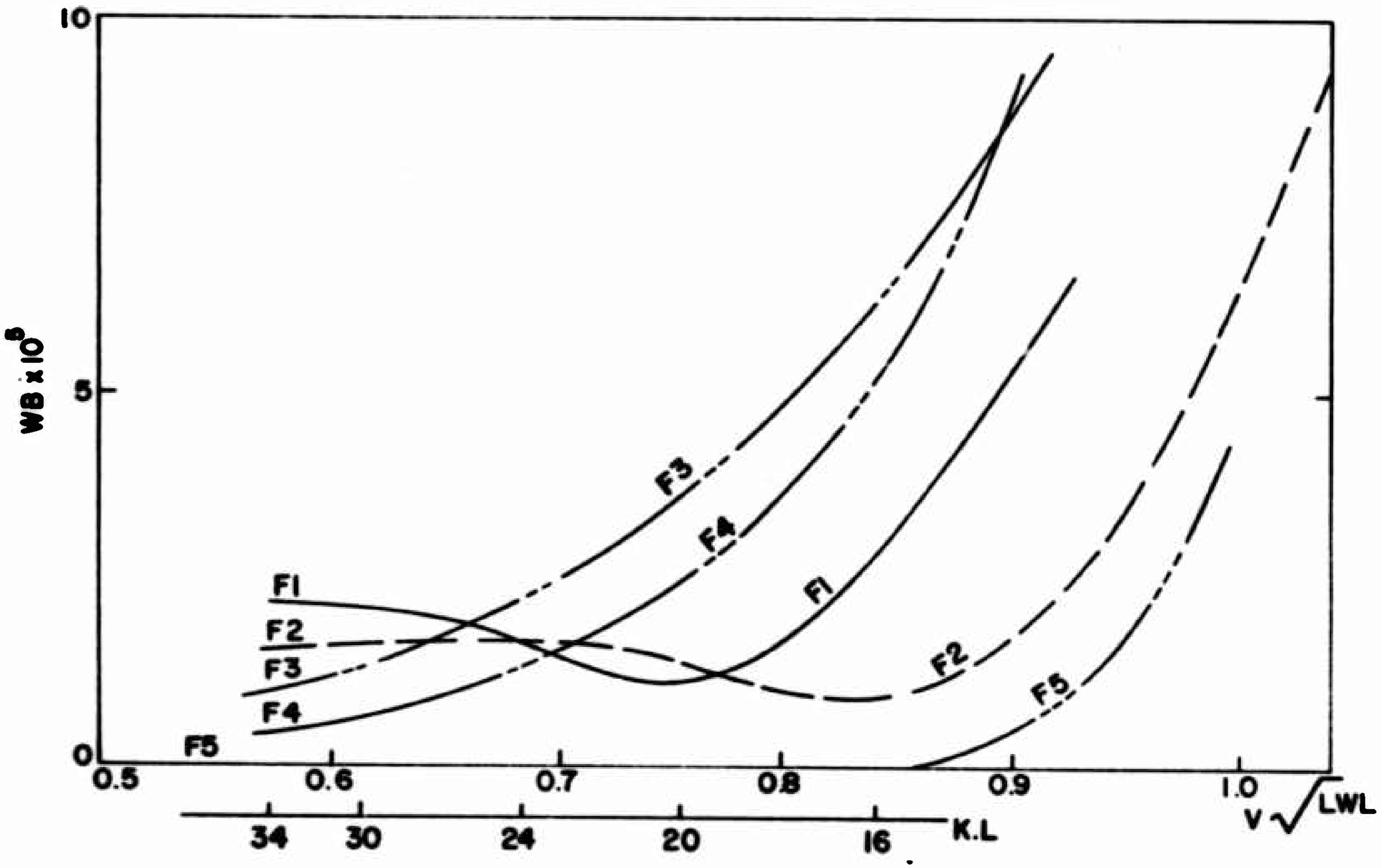


Figure 6. Curves of the fundamental term of the wave-making resistance due to a bow wave done in terms of $C_{WB} = R_{WB}/(\frac{1}{2}\rho U^2 L^2)$ for Series F models.

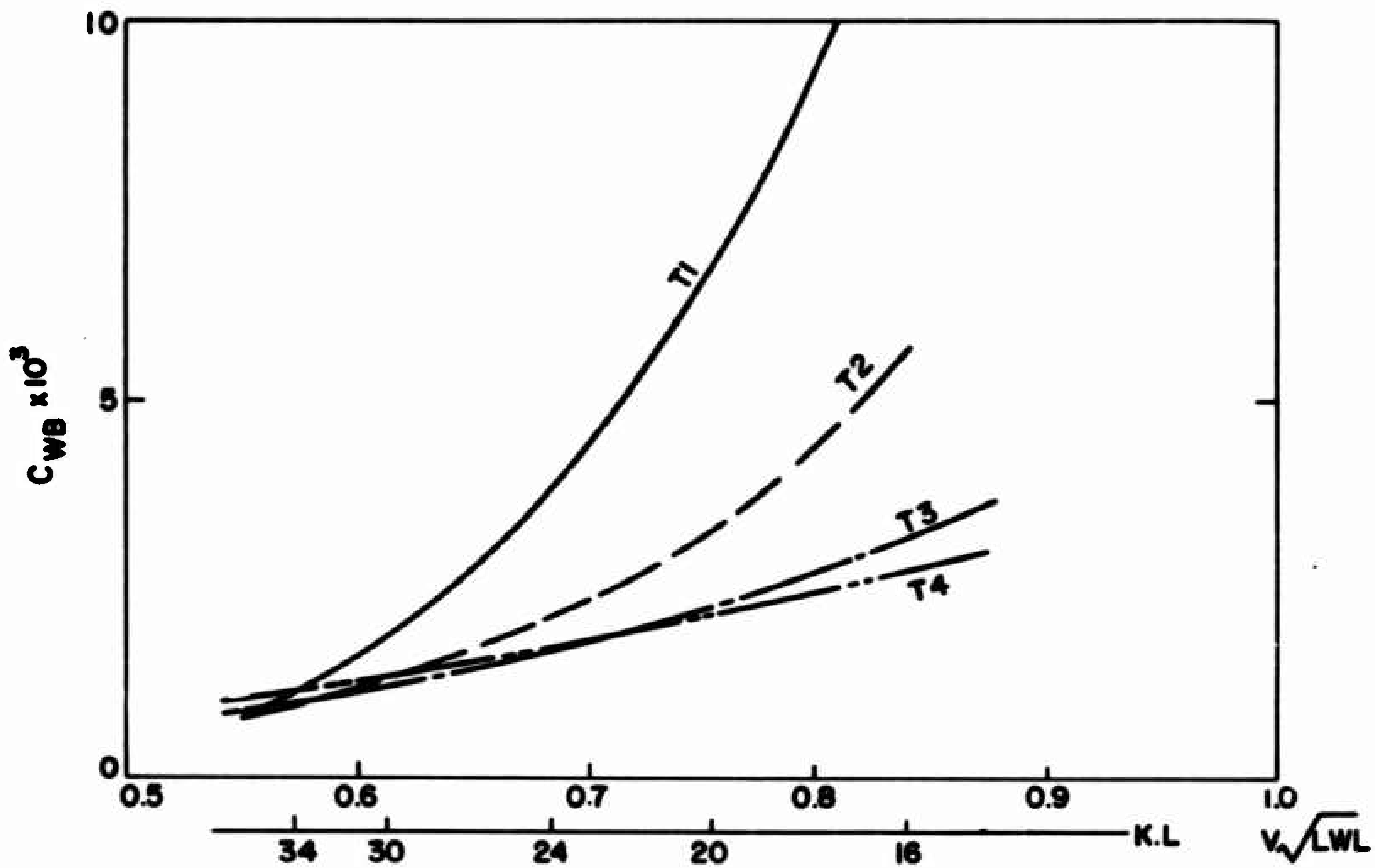


Figure 7. Curves of the fundamental term of the wave-making resistance due to a bow wave alone in terms of $C_{WB} = R_{WB}/(\frac{1}{2}\rho U^2 L^2)$ for Series T models.

The correlation between the singularity distributions of F1 and F2 (Figure 1) and their resistance curves (Figure 6) indicates the possibility of designing a ship so as to have minimum values in the monotonic term of resistance at a particular speed by putting an appropriate hollowness on the singularity distribution curve. In the vicinity of this hollow in the resistance curve the oscillating term is also damped, and the monotonic term of the resistance is considered to be nearly equal to the total resistance.

Model F5 is of particular interest from the stand point of minimum resistance. The singularity distribution of this model was determined so as to have a null magnitude of amplitude function in the whole speed range up to a speed length ratio of 0.8, retaining the identical amount of flux. The singularity distribution was concentrated close to midship and the corresponding hull form yielded an extremely fine entrance. The wave profile generated by this form at the speed of KoL = 20 (speed length ratio = 0.753) is shown in Figure 8. It has a distinguishable trough, which very likely causes a serious violation with the linearized wave theory. The calculated resistance results, therefore, should not be taken as they are.

A comparison of the wave profiles of F1 with that of the others resulted in an interesting phenomenon. It would be expected that the wave resistance of F1 would be the highest in terms of the wave profiles. Since the opposite is the case, further investigation in this area would seem to be indicated.

More information concerning boundary conditions yield singularity distributions which can be substituted for conventional ship forms is necessary before minimum wave resistance theory can be applied to practical ship design. The author has just begun work in this direction and does not have sufficient data to reach any definite conclusions. On the basis of the little data now available to him, however, he believes that further development in this direction will be of value.

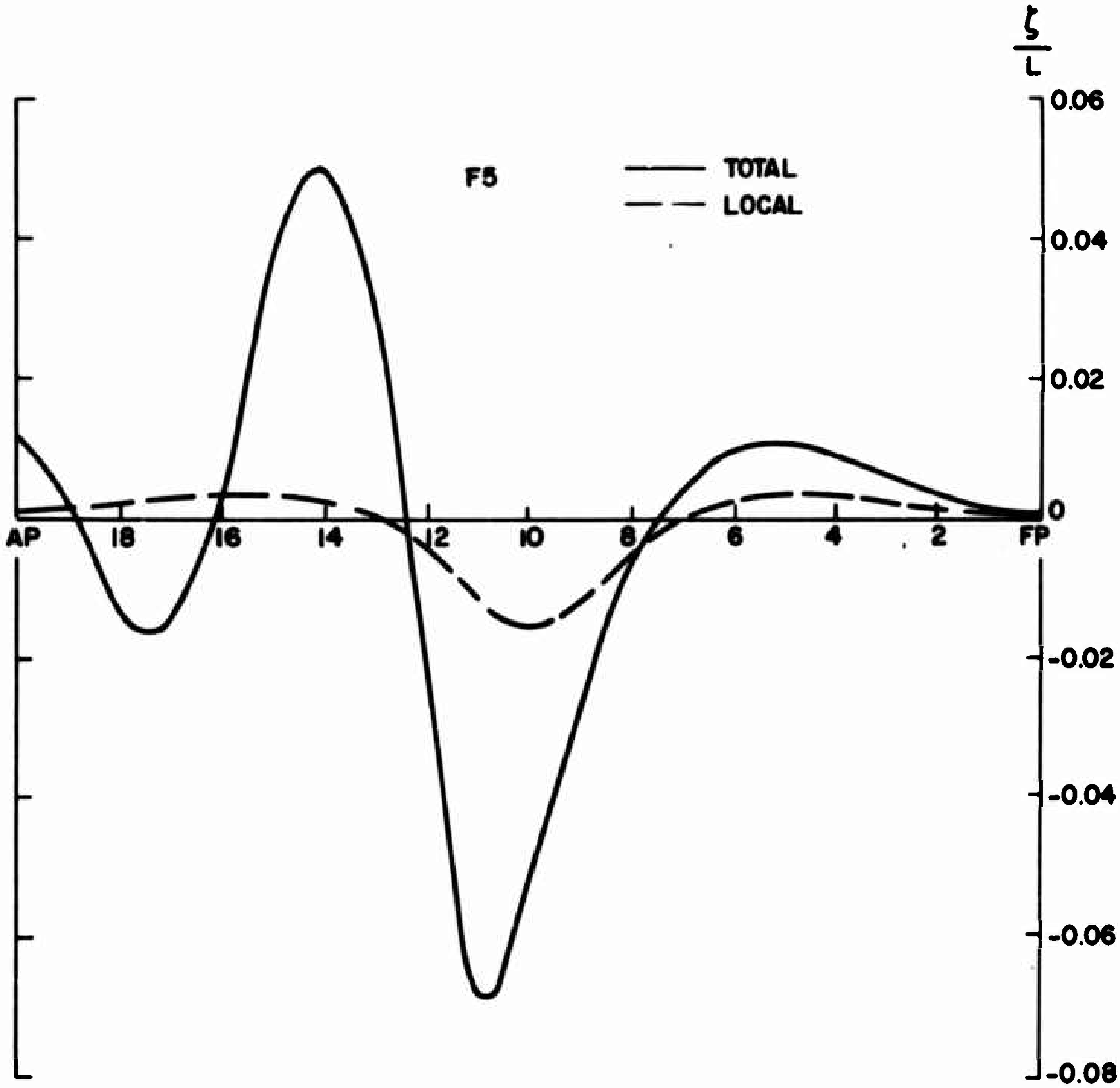


Figure 8. Calculated wave profiles of Model F5 at a speed of $K_0L = 20$ (speed length ratio = 0.753).

COMPARISON OF CALCULATED AND MEASURED WAVE PROFILES

None of the above mentioned models derived by means of stream line tracing has been constructed so far. Comparison of calculated and measured wave profiles has been done before only with the stream line traced models for simple forms of distribution. Agreement between measured and calculated wave profiles for such a model of a linear distribution, $m(\xi) = \xi$, with a deep draft (depth length ratio of the distribution = 0.25) was fairly satisfactory as seen in Figure 9.[5] The measured wave profiles for the same kind of models with a finite depth (depth length ratio = 0.05), C-201, also yielded good agreement except for a forward shift of the phase as a whole (Figure 10).[6,2] The phase shift with Model C-201 reached about six percent of model length at the speed of KoL = 14 or speed length ratio = 0.896. This amount decreased with lower speeds. C-201 was derived from a source distribution $m(\xi) = a_1 \sin(\frac{\pi}{2}\xi)$ with $a_1 = 0.6$, and the total flux out of for half model length was 87 percent of that of the models employed for wave profile calculation in this report.

Figure 11 is the measured wave profiles with a Series 60, a block coefficient of 0.70, parent model at speeds of KoL = 20 and 24, or speed length ratio = 0.753 and 0.685.

There appears to be no observable correlation between the measured wave profiles and the calculated wave profiles cited above. As can easily be imagined, a conventional hull is designed with more compound curvature than are the aforementioned models so as to attain a favorable wave cancellation, especially of the transverse wave component. Another serious effect comes out of the difference between a flat bottom of a conventional hull and the rocker keel of a mathematical model. The shoulder waves associated with the conventional ship also affects the wave profiles. Further investigation in this direction is necessary.

Figure 12 is the measured wave profiles of two 65,000 ton tanker models of a block coefficient of 0.81 at a speed of KoL = 30, or speed length ratio = 0.575. The after part of one of the models could not be covered by a photograph, because of the limited width of the towing tank.

The entrance of these models close to the bow is more similar to Model T1 than to any of the others. Both of the measured wave

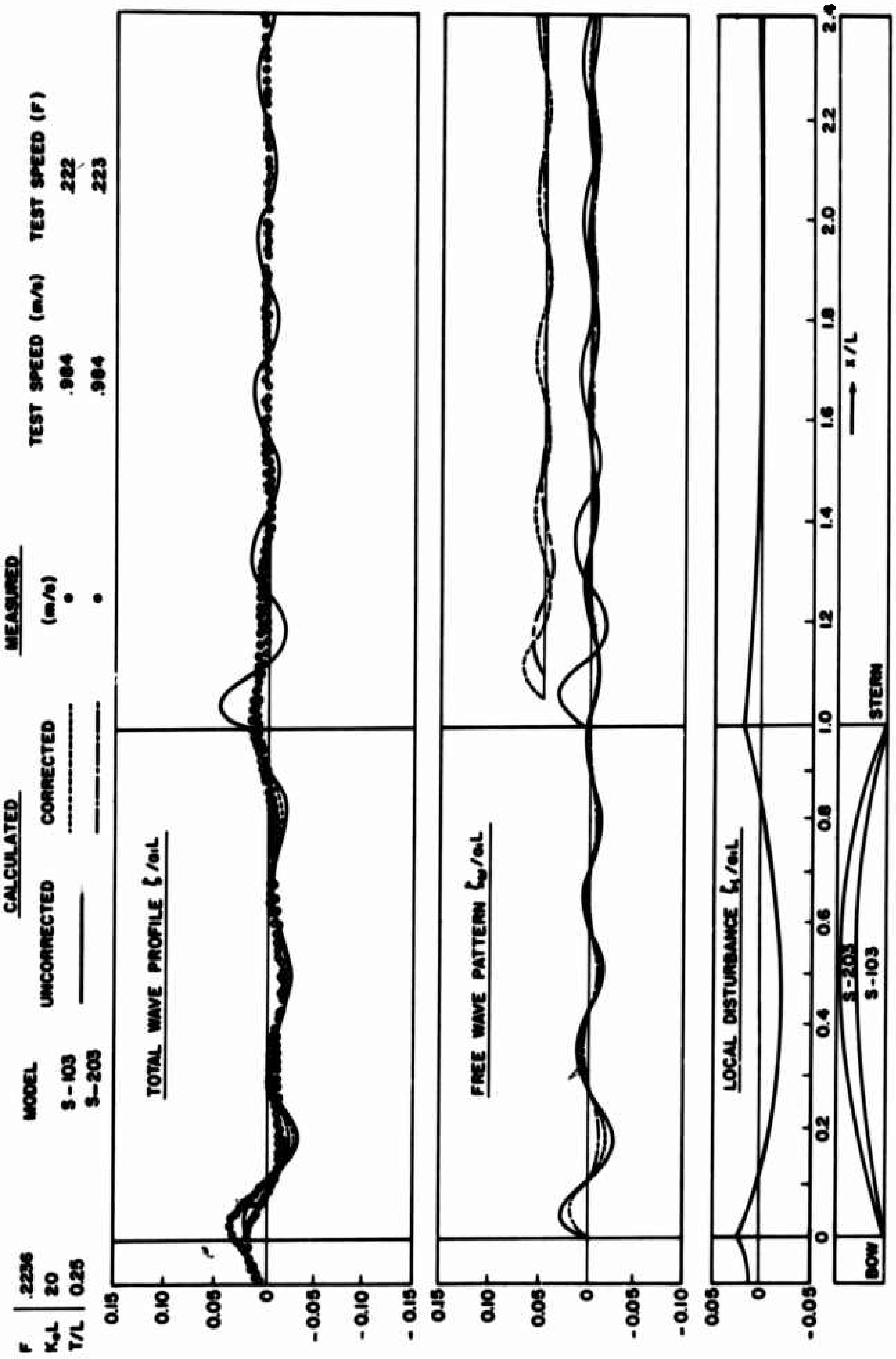


Figure 9. Measured wave profile with a deep draft model, compared with the calculated one.

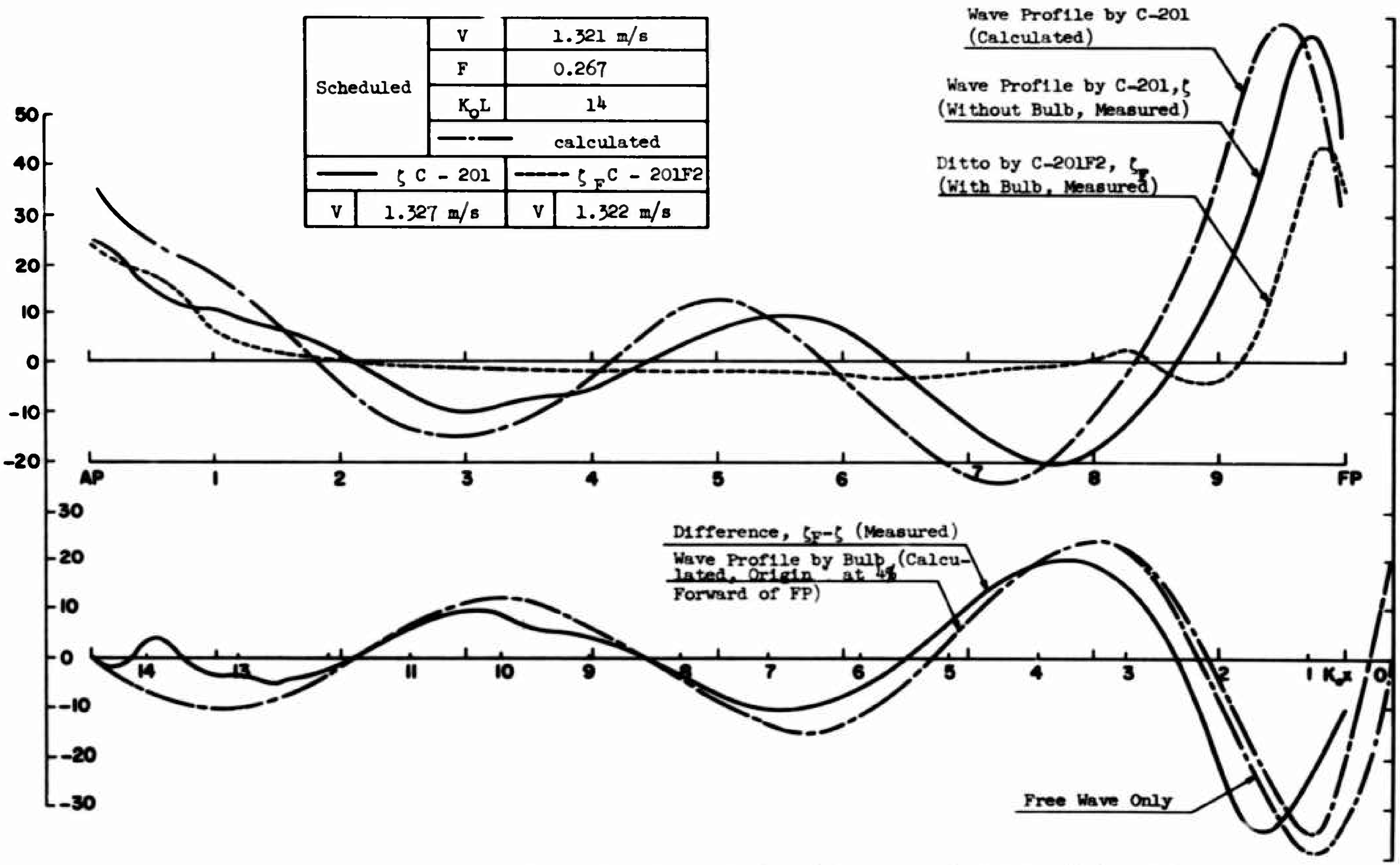


Figure 10. Measured wave profile with a finite draft model at a speed of KoL = 14 (speed length ratio = 0.896).

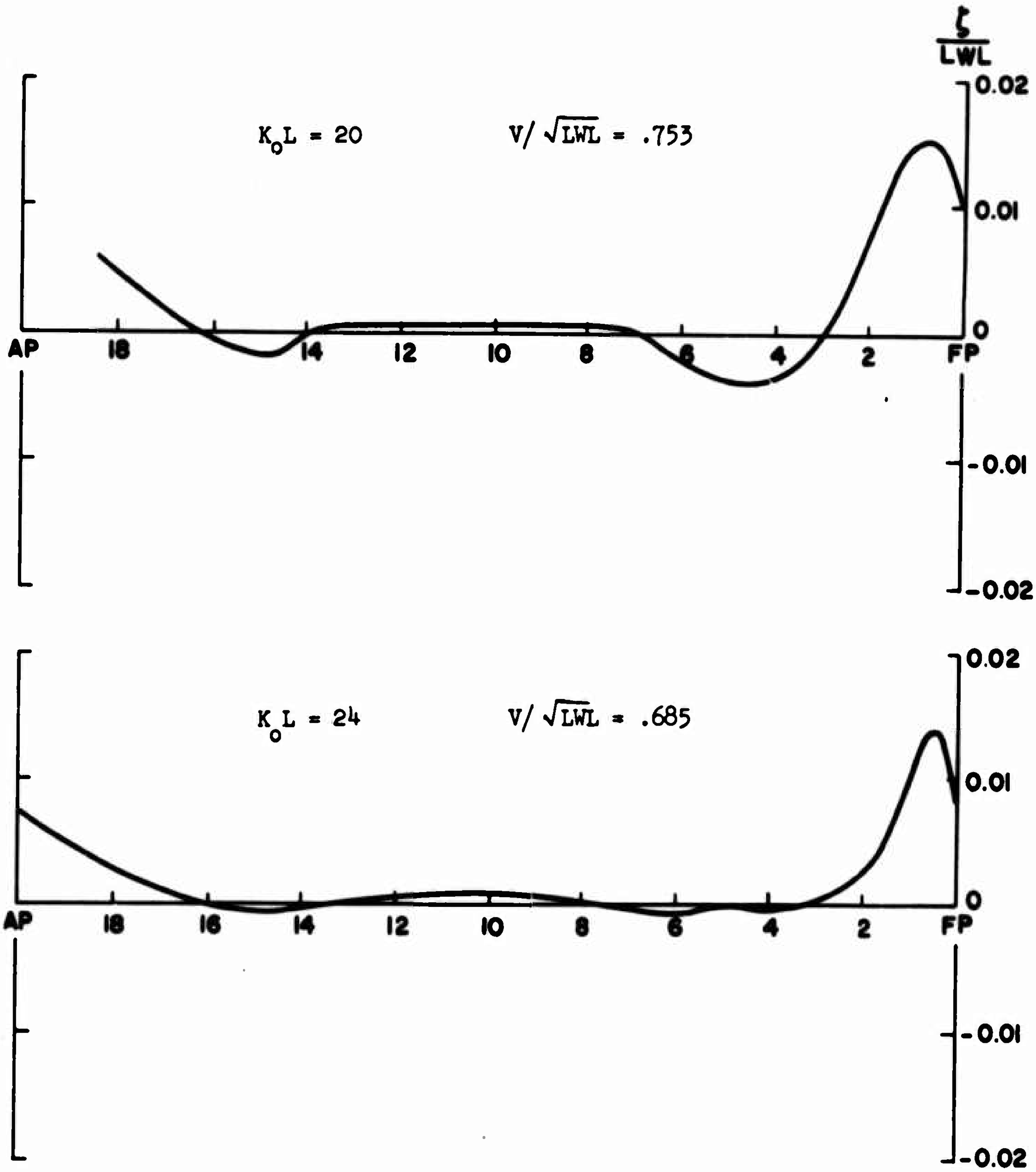


Figure 11. Measured wave profiles with a Series 60, C_B = 0.70, parent model (LWL = 12.71 ft.) at a speed of K_oL = 20 and 24 (speed length ratio = 0.753 and 0.685).

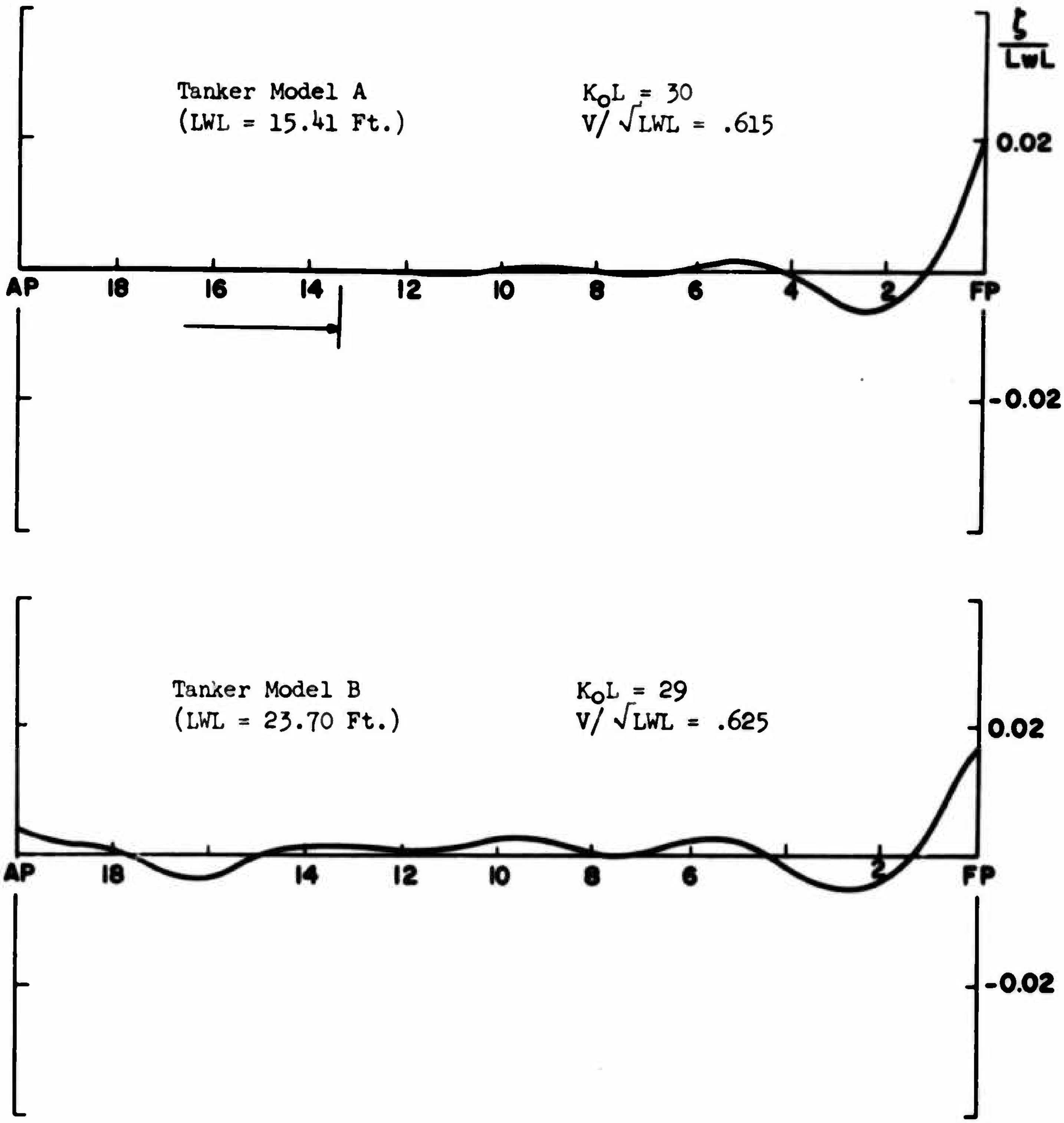


Figure 12. Measured profiles with two tanker models at a speed of K_oL = 30 (speed length ratio = 0.615).

profiles, however, are very small in their height compared to the one calculated for Model T1. This is believed to be due to the linearized surface condition as well as the results of the hull self-interference. Another cause of such a large difference is supposed due to the fact the wave profile is evaluated directly on the ships center plane where the singularity is located instead of on the ship's side. A further investigation on this point is in need.

CONCLUSIONS AND RECOMMENDATIONS

The systematic investigation of singularity distributions associated with stream line tracing to obtain a hull form which accurately meets the boundary conditions of the ship hull associated with an analysis of the wave profile alongside the model is expected to clarify the wave-making characteristics of ship as well as the mechanics through which the viscosity of a real fluid affects ship waves. An example of this procedure taking into consideration practical ship forms has been introduced.

The computer program to calculate wave profiles is now ready to be used for wave analysis. It is recommended that steps be taken to carry out analysis of wave profiles measured from mathematical models in order to check the validity of linearized boundary conditions which the wave theory has employed and the effect of viscosity on wave making phenomena.

The program has been made to take care of any depth of singularity distribution so that it will be easy to extend the program to figure out wave profiles along a ship which is to be substituted by a singularity distribution varied depthwise. With the varied singularity having a different form of function $m(\xi)$ at a different depth it should be more feasible to obtain a ship form closer to that of a conventional ship even from a singularity distributed over the ship central plane. In this paper the singularity distributions were represented by fifth order polynomials. It is the opinion of the author that this is insufficient for the representation of conventional ship forms, and that polynomial expressions of much higher order are necessary.

The use of higher order polynomials will facilitate obtaining optimum practical hulls because of the increase in the number of variable parameters in the polynomials.

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TABLE 1. SERIES F MODELS

Model No.	a_0	a_1	a_2	a_3	a_4	a_5	m_o
	m(ξ)						
F1	0	-3.36[illegible]5	26.3543	-43.5152	19.4821	0.4079	0.400
F2	0	2.4158	1.1472	-10.8371	9.0703	-1.5303	0.263
F3	0	2.381	-2.2516	0	0	0	0.129
F4	0	2.2653	-0.0087	-4.0298	2.5399	-0.1247	0.004
F5	0	18.8053	-71.5497	105.0588	-70.7719	18.4867	0.0

Remarks: $m(\xi) = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3 + a_4\xi^4 + a_5\xi^5$, symmetrical with respect to midship.
$m_o = m(\xi = 1)$, strength of singularity at the fore end of length
Ship length = 2
Depth of distribution = 5 percent of ship length.
Models F2 through F4 were obtained by adjusting the magnitude of the singularity so as to have a constant flux, retaining the original form of the functions shown in Figure 46 in the discussion in Reference 2.

TABLE 2. SERIES T MODELS

Model No.	a_0	a_1	a_2	a_3	a_4	a_5	m_o
	m(ξ)						
T1	0	-4.57[illegible]	64.052	-254.4249	374.5816	-179.582	0.15
T2	0	-0.2243	24.5469	-113.3865	171.4553	-81.1915	1.2
T3	0	0	5.34	-26.64	45.35	-22.4	1.6
T4	0	0.589	-1.998	0.848	4.42	-1.908	1.95

Remarks: See Table 1.

APPENDIX

FUNCTIONS CONNECTED WITH THE CALCULATION OF WAVE PROFILE AND WAVE-MAKING RESISTANCE

Definition and Characteristics

The definition of the functions cited here has been given by M. Bessho. He investigated then thoroughly. Herein they are shown in an abbreviated form for convenience.

$$P_n(x,t) = 1/2\,[O_n^{(1)}(x,t) - O_n^{(2)}(x,t)] \tag{1.1}$$

$$Q_n(x,t) = 1/2\,[O_n^{(1)}(x,t) + O_n^{(2)}(x,t)] \tag{1.2}$$

$$O_n^{(1)}(x,t) = \lim_{\mu\to+0} \frac{(-i)^n}{4\pi} \int_{-\pi}^{\pi} du \int_0^{\infty} \frac{e^{-Kt+iKx\cos u}\cos^{n+2}u}{K\cos u-1+\mu\, i\cos u}\, dK$$

$$O_n^{(2)}(x,t) = \lim_{\mu\to+0} \frac{i^n}{4\pi} \int_{-\pi}^{\pi} du \int_0^{\infty} \frac{e^{-Kt-iKx\cos u}\cos^{n+2}u}{K\cos^2 u-1+\mu\, i\cos u}\, dK \tag{1.3}$$

where $x,t > 0$ and n is integer.

When $x < 0$, we define as follows:

$$O_n^{(1)}(x,t) = (-)^n O_n^{(2)}(-x,t)$$
$$O_n^{(2)}(x,t) = (-)^n O_n^{(1)}(-x,t) \tag{1.4}$$

Differentiation:

$$\frac{\partial}{\partial x}(Q_n) = Q_{n-1} + q_{n-1} \qquad \frac{\partial}{\partial t}(Q_n) = Q_{n-2} + q_{n-2} \tag{1.5}$$

$$\frac{\partial}{\partial x} P_n = P_{n-1} \qquad \frac{\partial}{\partial t} P_n = P_{n-2}$$

$$\frac{\partial^2}{\partial x^2} Q_n = \frac{\partial}{\partial t} Q_n + \frac{\partial}{\partial x} q_{n-1} \qquad \frac{\partial^2}{\partial x^2} P_n = \frac{\partial}{\partial t} P_n \tag{1.6}$$

Recurrence Formula:

$$Q_n+(n-1)Q_{n-2} = x(Q_{n-1}+Q_{n-3}+q_{n-1}+q_{n-3}) + 2t(Q_{n-2}+Q_{n-4}+q_{n-2}+q_{n-4})$$
$$nP_n+(n-1)P_{n-2} = x(P_{n-1}+P_{n-3}) + 2t(P_{n-2}+P_{n-4}) \quad (1.7)$$

where

$$q_{-n}(x,t) = (-)^{n-1} \frac{(x/t)^{n-2}}{2\sqrt{x^2+t^2}} \qquad n \geq 2 \quad (1.8)$$

$$q_{-1}(x,t) = \frac{1}{2t} \left(\frac{t}{\sqrt{x^2+t^2}} - 1\right)$$

$$q_{2n+1}(x,t) = -\frac{x}{t}\, q_{2n+2}(x,t), \qquad q_{2n} = -\frac{x}{t}\, q_{2n+1}(x,t) + \frac{(-)^n}{2t} \frac{\Gamma(n+3/2)}{\sqrt{\pi}\,(n+1)!}$$

Integral Representation:

$$P_{2n}(x,t) = (-)^n \int_0^{\pi/2} e^{-t\sec^2 u} \sin(x \sec u) \cos^{2n} u\, du$$
$$P_{2n+1}(x,t) = (-)^{n+1} \int_0^{\pi/2} e^{-t\sec^2 u} \cos(x \sec u) \cos^{2n+1} u\, du \quad (1.9)$$

$$O_{-n}^{(1)}(x,t) = \frac{(-)^{n-1}}{2} \int_{L_1+L_2} e^{tv^2-xv} \frac{v^{n-1}}{\sqrt{1+v^2}}\, dv, \qquad n \geq 1 \quad (1.10)$$

$$O_0^{(1)}(x,t) = -\frac{1}{2} \int_{L_1+L_2} e^{tv^2-xv} \left(\frac{1}{\sqrt{1+v^2}} - 1\right) \frac{dv}{v} \quad (1.11)$$

where L_1 and L_2 are the paths shown in Figure A-1.

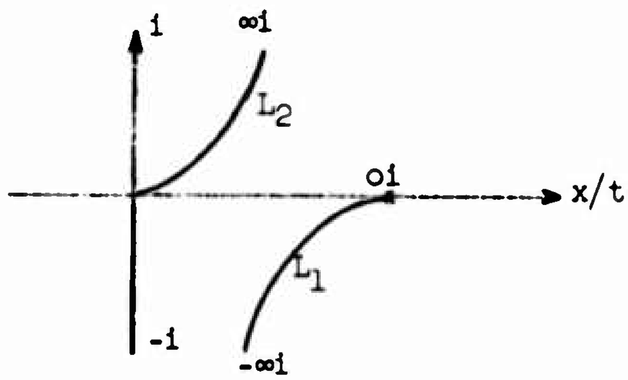

Figure A-1.

It follows that

$$Q_{-2n}(x,t) = \frac{1}{2}\int_0^{x/t} e^{tu^2-xu} \frac{u^{2n-1}du}{\sqrt{1+u^2}} + (-)^{n-1}\int_0^1 e^{-tu^2}\frac{\cos(xu)}{\sqrt{1-u^2}}u^{2n+1}du, \qquad n \geq 1$$

$$Q_{-2n-1}(x,t) = -\frac{1}{2}\int_0^{x/t} e^{tu^2-xu}\frac{u^{2n}}{\sqrt{1+u^2}}du + (-)^n\int_0^1 e^{-tu^2}\frac{\sin(xu)}{\sqrt{1-u^2}}u^{2n}du, \qquad x \geq 0 \qquad (1.12)$$

$$Q_o(x,t) = Q_o(0,t) - \frac{1}{2}\log\left(\frac{t+\sqrt{x^2+t^2}}{2t}\right) + \frac{1}{2}\int_0^{x/t}(e^{tu^2-xu}-1)\frac{du}{u\sqrt{1+u^2}} + \int_0^1 e^{-tu^2}(1-\cos xu)\frac{du}{u\sqrt{1-u^2}} \qquad (1.13)$$

$t \rightarrow 0$:

$\bar{P}_n$, $\bar{Q}_n$ as defined by T. H. Havelock

$$P_n(x,0) = \bar{P}_n(x) \qquad (1.14)$$

$$P_n(x,0) = -\frac{\pi}{2}\int_\infty^x \cdots \int_\infty^x Y_o(x)(dx)^{n+1}$$

$$P_{-n}(x,0) = -\frac{\pi}{2}\left(\frac{d}{dx}\right)^{n-1} Y_o(x) \qquad (1.15)$$

$$\bar{Q}_{-1}(x) = \frac{\pi}{2}\left[\bar{H}_o(x) - Y_o(x)\right] = 2\,0_{-1}^{(1)}(x,0)$$

$$\bar{Q}_{-1}(x) = \int_0^x \bar{Q}_{n-1}(x)\,dx \qquad (1.16)$$

$$\bar{Q}_o(x) = \log 2\gamma x + 2\,0_o^{(1)}(x,0)$$

$$\bar{Q}_1(x) = 1 - x + x\log 2\gamma x + 2\,0_1^{(1)}(x,0) \qquad (1.17)$$

$$\bar{Q}_2(x) = 1/4 + x - 2/4\,x^2 + \frac{(x^2-1)}{2}\log 2\gamma x + 2\,0_2^{(1)}(x,0)$$

where Y_o is a Bessel function of the second kind, H_n is a Struve function, and the numerical tables are available in the Theory of Bessel Functions by Watson.

$\bar{P}_n, \bar{Q}_n$: T.H. Havelock, Proc. Roy. Soc., 103 (1923), 108 (1921), 135 (1939)
W.C.S. Wigley, Proc. Roy. Soc., 144 (1943)

log γ = C is the Enler Constant (C = 0.577215 ...)

$x \to 0$

$$P_{2n}(0,t) = 0 \;, \qquad P_{-2n-1}(0,t) = (-)^n U_n(t) \tag{1.18}$$

$$\begin{aligned} U_o(t) &= \frac{1}{2} e^{-\frac{t}{2}} K_o\left(\frac{t}{2}\right) \\ U_1(t) &= \frac{1}{4} e^{-\frac{t}{2}} \left[K_o\left(\frac{t}{2}\right) + K_1\left(\frac{t}{2}\right)\right] \\ U_{n+1}(t) &= \left(1 + \frac{n}{t}\right) U_n(t) - \frac{(n - 1/2)}{t} U_{n-1}(t) \end{aligned} \tag{1.19}$$

where

$$U_n(t) = \frac{e^{-t}}{2} V_{-n}(t) \qquad\qquad U_n(t) = \frac{e^{-t}}{2} V_o$$

and

$$V_n(t) = \sqrt{\pi}\, t^{\frac{n-1}{2}} e^{\frac{t}{2}} W_{-\frac{n}{2},-\frac{n}{2}}(t) = \sqrt{\pi}\, t^{\frac{n-1}{2}} e^{\frac{t}{2}} W_{-\frac{n}{2},\frac{n}{2}}(t)$$

and K_n is a modified Bessel function of the second kind (according to Watson or McLachlan), and

$$W_{K,\mu}(t) = \frac{t^{\frac{n+1}{2}}}{\sqrt{\pi}} \int_0^\infty e^{-\frac{t}{2}\cosh u} \sinh^{2(\mu-K)} \frac{u}{2} \cosh^{2(\mu+K)} \frac{u}{2}\, du$$

is a Whittaker function (see Mod. Ar lyses, p. 388).

$$O_{2n+1}(0,t) = P_{2n+1}(0,t) \;, \qquad O^{(1)}_{2n}(0,t) = Q_{2n}(0,t) \tag{1.20}$$

$$Q_{-2n-2}(0,t) = (-)^n E_n(t) \qquad n \geq 0$$

$$E_o(t) = e^{-t} \int_0^1 e^{tv^2} dv = \int_0^1 e^{-tu^2} \frac{udu}{\sqrt{1-u^2}} = \sum_{n=0}^{\infty} \frac{(-t)^n}{(\frac{3}{2})_n} \tag{1.21}$$

$$E_1(t) = \left(1 + \frac{1}{2t}\right) E_o(t) - \frac{1}{2t}$$

$$\frac{d}{dt} E_n(t) = - E_{n+1}(t) , \qquad E_n(0) = \frac{n!}{(\frac{3}{2})_n}$$

$$E_{n+1}(t) = (1 + \frac{n + \frac{1}{2}}{t}) E_n(t) - \frac{n}{t} E_{n-1} (t) \tag{1.22}$$

$$Q_o(0,t) = - \frac{1}{2} \log (4\gamma t) + \int_0^t E_o(t)\, dt$$

where

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = \alpha(\alpha+1) \cdots (\alpha+n-1)$$

$$(\alpha)_n = 0 , \qquad \alpha < 0$$

$$(\alpha)_o = 1$$

Expanding Equation (1.9) in harmonic functions and integrating term by term,

$$P_{2n}(x,t) = \sum_{m=0}^{\infty} \frac{x^{2m+1}}{(2m+1)!} P_{2n-2m+1}(0,t) = \sum_{m=0}^{\infty} \frac{(-)^{m-n} x^{2m+1}}{(2m+1)\,!} U_{m-n}(t) \tag{1.24}$$

$$P_{2n+1}(x,t) = \sum_{m=0}^{\infty} \frac{x^{2m}}{(2m)!} P_{2n-2m+1}(0,t) = \sum_{m=0}^{\infty} (-)^{m-n+1} \frac{x^{2m}}{(2m)!} U_{m-n+1}(t) \tag{1.25}$$

$$Q_{-n}(x,t) = R_{-n}(x,t) + S_{-n} (x,t) , \qquad n \geq 1 \tag{1.26}$$

$$R_{-n}(x,t) = \frac{(-)^n}{2} \int_2^{\frac{x}{t}} e^{(t u^2 - xu)} \frac{u^{n-1}}{\sqrt{1 + u^2}} du, \qquad n \geq 1 \tag{1.27}$$

$$S_{-2n}(x,t) = (-)^{n-1} \int_0^{\frac{\pi}{2}} e^{-t \cos^2 u} \cos(x \cos u) \cos^{2n-1} u\, du, \quad n \geq 1$$

$$S_{-2n-1}(x,t) = (-)^n \int_0^{\frac{\pi}{2}} e^{-t \cos^2 u} \sin(x \cos u) \cos^{2n} u\, du, \quad n \geq 0 \tag{1.28}$$

$$S_{-2n}(x,t) = (-)^{n-1} \sum_{m=0}^{\infty} \frac{(-)^m x^{2m}}{(2m)!} E_{n+m-1} (t) , \qquad n \geq 1$$

$$S_{-2n-1}(x,t) = (-)^n \sum_{m=0}^{\infty} \frac{(-)^m x^{2m+1}}{(2m+1)!} E_{n+m} (t) , \qquad n \geq 0 \tag{1.29}$$

$$S_{-n}(x,t) = \sum_{m=0}^{\infty} \frac{t^m}{m!} S_{-n-2m}(x,0) , \qquad n \geq 1 \tag{1.30}$$

$$S_{-1}(x,t) = \frac{\pi}{2} \overrightarrow{H_o}(x) , \qquad S_{-2}(x,0) = \frac{\pi}{2} \overline{H_o'}(x) \tag{1.31}$$

Asymptotic Expansion:

$$P_{-1}(x,t) \underset{t>>x}{\approx} -\frac{1}{2}\sqrt{\frac{\pi x}{2t}}\, e^{-t} \sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)}{n!}\left(-\frac{x}{2t}\right)^n Y_{n+1/2}(x) \tag{1.32}$$

$$P_{-1}(x,t) \underset{x>>t}{\approx} -\frac{\sqrt{\pi}}{2} e^{-t} \sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)}{n!} \left(\frac{2t}{x}\right)^n Y_n(x) \tag{1.33}$$

By differentiation and integration of the above equation, we have

$$P_{-\nu}(x,t) \approx -\frac{\sqrt{\pi}}{2} e^{-t} \sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)}{n!} (2t)^n \left(\frac{d}{dx}\right)^{\nu-1} \left|\frac{Y_n(x)}{x^n}\right| , \quad \nu \geq 1$$

(1.34)

$$P_{\nu}(x,t) \approx -\frac{\sqrt{\pi}}{2} e^{-t} \sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)}{n!} (2t)^n \int_{\infty}^{x} \cdots \int_{\infty}^{x} (dx)^{\nu+1} \frac{Y_n(x)}{x^n} , \quad \nu \geq 1$$

where

$$|x| > | , \qquad |x| >> |n| \qquad \text{and} \qquad -\frac{1}{2}\pi \leq \text{phase } x \leq \frac{1}{2}\pi$$

$$Y_n(x) \approx \sqrt{\frac{2}{\pi x}} \left|\zeta_n(x) \sin\psi + \xi_n(x)\cos\psi\right|$$

where

$$\zeta_n(x) \approx 1 - \frac{4(n^2-1^2)(4n^2-3^2)}{2!(8x)^2} + \frac{4(n^2-1^2)(4n^2-3^2)(4n^2-5^2)(4n^2-7^2)}{4!(8x)^4} - \cdots$$

$$\xi_n(x) \approx \frac{(4n^2-1^2)}{1!(8x)} - \frac{(4n^2-1^2)(4n^2-3^2)(4n^2-5^2)}{3!(8x)^3} + \cdots$$

$$x = \left(x - \frac{1}{4}\pi - \frac{1}{2}n\pi\right).$$

(see McLachlan; Bessel Function for Engineers)

For example

$$Y_0(x) \approx \sqrt{\frac{2}{\pi x}} \left[\sin(x - \frac{\pi}{4}) - \frac{1}{8x} \sin(x + \frac{\pi}{4})\right]$$

$$Y_1(x) \approx \sqrt{\frac{2}{\pi x}} \left[-\sin(x + \frac{\pi}{4}) + \frac{3}{8x} \sin(x - \frac{\pi}{4})\right]$$

or more approximately

$$Y_0(x) \approx \sqrt{\frac{2}{\pi x}} \sin(x - \frac{\pi}{4})$$

$$Y_1(x) \approx \sqrt{\frac{2}{\pi x}} \left[-\sin(x + \frac{\pi}{4})\right]$$

Mathematical Table:

(1) $\bar{P}_n(x) = P_n(x,0)$, $n = 0(1)9$, $x = 0(0,4)4.4$, $5.0(1)40$

4 digits, JINNAKA; SNA of JAPAN, Vol. 84, where ( ) indicates the increment between table values.

(2) $\frac{2}{\pi}\bar{P}_0(x) = -\int_0^x Y_0(x)dx$

$\frac{2}{\pi}[\bar{P}_1(x) - 1] = \frac{2}{\pi}\int_0^x P_0(x)dx$ $\quad x = 0(0.1)1.0(0.2)10(0.4)50$

given to six significant figures

(3) $\frac{1}{2\pi}\bar{Q}_0(x) = \frac{1}{4}\int_0^x \bar{H}_0(x) - Y_0(x)\, dx$

$\frac{1}{2\pi}\bar{Q}_1(x) = \frac{1}{2\pi}\int_0^x \bar{\bar{Q}}_0(x)\, dx$

$x = 0(0.08)0.48, 0.64, 0.8, 1.0(0.4)15.4, 16(2), 32(4), 44, 50$, given to four through six significant numbers.

(4) $-P_{-3}(x,t)$, N.P.L.M_a/16/1502, where x and t are represented by β $(=x)$ and α $(=\sqrt{t})$ respectively.

(5) Integration of Equation (2.3) for wave elevation by point source. B.V. Korvin-Kronkovski and others; SNAME Technical and Research Bulletin No. 1-16 (1954).

(6) $Q_{-1}^{(1)}(x,t)$ and $P_{-1}(x,t)$
The Wave-Making Subcommittee of the Japan Towing Tank Committee. The University of Michigan.

Application of the Functions

(1) The velocity potential at $P(\xi, \eta, \zeta)$ by a point-source of unit strength located at $Q(\xi', \eta', \zeta')$ for steady motion in deep water with a uniform velocity of U coming on in the negative direction of the ξ-axis.

According to the definition as $u = -\frac{\partial\varphi}{\partial x}$ etc., the velocity potential is

$$\phi = \frac{1}{4\pi} S(P, Q) + U\xi$$

$$S(P,Q) = \frac{1}{r(P,Q)} + \frac{1}{r(P,\bar{Q})} - \frac{1}{\pi}\frac{\partial}{\partial\zeta}\int_{-\pi}^{\pi} \epsilon\left[K_o \sec^2\theta\right.$$

$$\left.[(\zeta+\zeta') + i(\xi-\xi')\cos\theta + i(\eta-\eta')\sin\theta]\right] d\theta \qquad (2.1)$$

where

$$\epsilon(\alpha,\beta) = \int_0^\infty \frac{e^{K\beta}\,dK}{K-\alpha+\mu i\sec\theta} \qquad (2.2)$$

and $\bar{Q}$ = the image point of Q.
As for the table of $\epsilon(\alpha, \beta)$, it is referred to as N.B.S. mathematical tables No. 51 (1958).

On the ship's centerplane, S(P,Q) can be reduced as follows:

$$S(P,Q)\Big|_{\eta=\eta'=0} = \frac{1}{\sqrt{(\xi-\xi')^2 \quad (\zeta-\zeta')^2}} - \frac{1}{\sqrt{(\xi-\xi')^2 \quad (\zeta-\zeta')^2}}$$

$$+ 4K_o\, O_{-1}^{(1)}(K_o\overline{\xi-\xi'}, -K_o\overline{\zeta+\zeta'}) \qquad (2.3)$$

where $K_o = g/v^2$.

(2) The surface elevation, ζ_s, at a point P(ξ, 0, 0) due to a point source located at Q(ξ', 0, ζ') with a uniform stream of U in the negative ξ-direction.

$$4\pi K_o\zeta_s(\xi,Q) = -\frac{1}{U}\frac{\partial}{\partial\xi} S(\xi, Q)$$

$$= -\frac{4K_o^2}{U}\left[O_{-3}^{(1)}(K_o\overline{\xi-\xi'}, -K_o\zeta') + q_{-3}(K_o\overline{\xi-\xi'}, -K_o\zeta')\right]$$

$$(2.4)$$

Putting $x = K_o\xi$, $x' = K_o\xi'$ and $t = -K_o\xi'$, the surface elevation is

$$\zeta_s(P,Q) = -\frac{K_o}{\pi U}\left[O_{-3}^{(1)}(x-x',t) + q_{-3}(x-x',t)\right]$$

$$= -\frac{K_o}{\pi U}\frac{\partial}{\partial t} O_{-1}^{(1)}(x-x',t) \qquad (2.5)$$

(3) The surface elevation, ζ_D, at a point $P(\xi, 0, 0)$ due to a unit doublet* located at $Q(\xi', 0, \zeta')$ with a uniform stream of U in the negative ξ-direction.

Putting $x = K_o\xi$, $x' = K_o\xi'$ and $t = -K_o\zeta'$
ζ_D can be written in the form

$$\zeta_D(P,Q) = -\frac{\partial}{\partial\xi}\,\zeta_s(\xi,Q) = -\frac{K_o}{\partial x}\,\zeta_s(x, Q)$$

$$= \frac{K_o^2}{\pi U}\left[(O_{-4}^{(1)}(x-x',t) + \frac{t}{2(x^2+t^2)^{3/2}} - \frac{(x/t)^2}{2(x^2+t^2)^{1/2}}\right] \quad (2.6)$$

In the rear of the double

$$\zeta_D(x,t) = \frac{K_o^2}{\pi U}\left[O_{-4}^{(1)}(x-x',t) - 2P_{-4}(x'-x,t)\right.$$

$$\left. + \frac{t}{2(x^2+t^2)^{3/2}} - \frac{(x/t)^2}{2(x^2+t^2)^{1/2}}\right] \quad (2.7)$$

The term of $-2P_{-4}(x'-x,t)$ pertains to the free travelling wave component.

(4) The surface elevation due to a linear lengthwise and uniform draftwise distribution.**

$$\begin{aligned} M_1(\xi') &= a_1 U\xi' & -\ell \leq \xi' \leq \ell \\ M_2(\zeta') &= 1 & -T \leq \zeta' \leq 0 \end{aligned} \quad (2.8)$$

Using the parameter x and x' for $K_o\xi$ and $K_o\xi'$ respectively, the surface elevation is given in the form

$$\zeta(x) = \frac{4a_1}{K_o^2}\int_0^t dt \int_{-\frac{x_o}{2}}^{x_o/2} x'dx'\left[\frac{\partial}{\partial t}\,O_{-1}^{(1)}(x-x',t)\right]$$

$$= \frac{4a_1}{K_o^2}\int_{-\frac{x_o}{2}}^{x_o/2} \left| O_{-1}^{(1)}(x-x',t)\right|_{t=o}^{t=t_o} x'dx' . \quad (2.9)$$

*The strength of the doublet μ is related to a sphere of the radius of a placed in an infinite uniform flow as $\mu = 2\pi a^3 U$.

**According to Michell's approximation, this distribution indicates a ship of $L = 2\ell$ length, $B = 2b$ breadth and T draft with $a_1 = b/\pi\ell^2 = 2B/(\pi L^2)$. For the more accurate ship form, see the model in the contents.

or $\zeta(x) = Z_o + Z_1$ (2.10)

$$Z_o = \frac{4a_1}{K_o^2} \left| \left| (x'-x) \int^{x'} q_1 dx' - \iint^{x'} q_1 dx'^2 - \int^{x'} q_1 dx' \right|_{x'=\beta}^{x'=\alpha} \right|_{t=o}^{t=t_o}$$

$$= \frac{4a_1}{K_o^2} \left[\frac{x}{2} \log \frac{\beta(t_o + \sqrt{t^2 + \alpha^2})}{\alpha(t_o \sqrt{t^2 + \alpha^2})} - \frac{t_o}{2} \log \left[\frac{\beta + \sqrt{t_o^2 + \beta^2}}{\alpha + \sqrt{t_o^2 + \alpha^2}} + \frac{\sqrt{t_o^2 + \beta^2} - t_o}{2\beta} \right.\right.$$

$$\left.\left. - \frac{\sqrt{t^2 + \alpha^2} - t_o}{2\alpha} \right] \right] \quad (2.11)$$

where

$$\alpha = x + \frac{x_o}{2} , \qquad \beta = x - \frac{x_o}{2}$$

and

$$Z_1 = \frac{4a_1}{K_o^2} \left| \left| (x-x') O_o^{(1)}(x',t) + O_{-1}^{(1)}(x',t) \right|_{x'=\beta}^{x'=\alpha} \right|_{t=o}^{t=t_o}$$

$$= \frac{4a_1}{K_o^2} \left| O_1^{(1)}(\alpha,t) - O_1^{(1)}(\beta,t) - \lambda \left[O_o^{(1)}(\alpha,t) + O_o^{(1)}(\beta,t) \right] \right|_{t=o}^{t=t_o}$$

(2.12)

while Z_o is symmetrical with respect to the midship, Z_1 contains the oscillating term. $O_n^{(1)}(x,t)$ is monotonic for $x > 0$, but for $x < 0$, from Equations (1.1) and (1.4),

$$O_n^{(1)}(x,t) = (-)^n \left[O_n^{(1)}(-x,t) - 2P_n(-x,t) \right] \quad (2.13)$$

where $P_n(-x,t)$ is the oscillating term.

(i) $\alpha > 0$, $\beta > 0$, i.e., forward of the F. P., we can use Z_1 as it is in Equation (2.12).

(ii) $\alpha > 0$, $\beta < 0$, i.e., within the ship's length

$$Z_1 = \frac{4a_1}{K_o^2} \Big| O_1^{(1)}(\alpha,t) + O_1^{(1)}(-\beta,t) - \lambda \Big[O_o^{(1)}(\alpha,t) + O_o^{(1)}(-\beta,t) \Big]$$

$$+ \; 2\lambda \, P_o(-\beta,t) - 2P_1(-\beta,t) \Big|_{t=o}^{t=\tau} \qquad (2.14)$$

(iii) $\alpha < 0$, $\beta < 0$, i.e., aftward of the AP

$$Z_1 = \frac{4a_1}{K_o^2} \Big| -O_1^{(1)}(-\alpha,t) - O_1^{(1)}(-\beta,t) - \frac{x_o}{2} \Big[O_o^{(1)}(-\alpha,t) + O_o^{(1)}(-\beta,t) \Big]$$

$$+ \; x_o \Big[P_o(-\alpha,t) + P_o(-\beta,t) \Big] + 2 \Big[P_1(-\alpha,t) - P_1(-\beta,t) \Big] \Big|_{t=o}^{t=t_o}$$

(2.15)

(5) A linear distribution with respect to the draft

$$m_1(\xi) \qquad\qquad -\ell \le \xi \le \ell$$

$$m_2(\zeta) = (1 + \zeta/T) \qquad -T \le \zeta \le 0 \qquad (2.16)$$

The surface elevation along the ship center line in this case, ζ_V, denoting the elevation due to the uniform distribution by ζ_U, is given as follows

$$\zeta_V\,(x,0,0) = \zeta_U(x,0,0)$$

$$+ \frac{1}{t} \int_{-\frac{x_o}{2}}^{x_o} m_1(x') \Big[t_o O_{-1}^{(1)}(x-x';t_o) + O_{-1}^{(1)}(x-x',0)$$

$$- \; O_1^{(1)}(x-x',t_o) + \frac{t_o - \sqrt{(x-x')^2 + t_o^2}}{2(x-x')} \Big] dx'$$

$$- \frac{1}{t_o} \int_{x}^{x_o/2} m_1(x') \Big[t \, O_{-1}^{(1)}(x'-x,t_o) - O_1^{(1)}(x'-x,0)$$

$$-O_1^{(1)}(x'-x,t)$$

$$- 2t_o P_{-1}(x'-x,t) - 2P_1(x'-x,0) + 2P_1(x'-x,t_o)$$

$$\left. + \frac{t_o - \sqrt{(x'-x)^2 + t_o^2}}{2(x'-x)} \right] dx' \qquad (2.17)$$

where $x = K_o\xi$, $x' = K_o\xi'$ and $t_o = K_oT$

(6) Wave-making resistance by a source distribution on a vertical central plane

$$R = \frac{\rho K_\Omega^2}{\pi} \int_0^{\pi/2} |F(K,\Theta)|^2 \sec^3\Theta \, d\Theta \qquad (2.18)$$

$$F(K,\Theta) = \iint m(\xi',\zeta') \, e^{Kz - iKx\cos\Theta} \, d\xi' d\zeta' \qquad (2.19)$$

where $K = K_o \sec^2$
Hence,

$$R = -\frac{\rho K_\Omega^2}{\pi} \iint \sigma(\xi,\zeta) d\xi d\zeta \iint \sigma(\xi',\zeta') d\xi' d\zeta' P_{-3}(K_o\overline{\xi-\xi'}, \; K_o\overline{\zeta+\zeta'})$$

$$(2.20)$$

For the linear source distribution such as in Equation (2.8)

$$R(K,\Theta) = \frac{2a_1U}{iK^3\cos^2\Theta} (1-e^{-Kt})[\sin(K\ell\cos\Theta) - K\ell\cos\Theta \cos(K\ell\cos\Theta)]$$

$$(2.21)$$

or

$$F(K_o\sec^2\Theta, \; \Theta) = \frac{2a_1U}{iK_o^3} \cos^4\Theta(1-e^{-t_o\sec^2\Theta})$$

$$\times [\sin (\frac{x_o}{2} \sec\Theta) - \frac{x_o}{2} \sec\Theta \cos (\frac{x_o}{2} \sec\Theta)] \qquad (2.22)$$

where $t_o = K_oT$, $x_o = K_oL$

Hence,

$$|F(K_o \sec\theta,\theta)|^2 = \frac{4a_1^2U^3}{K_o^6}\cos^8\theta(1-2e^{-t_o\sec^2\theta}+e^{-2t_o\sec^2\theta})$$

$$\times\left[\frac{1}{2}+\frac{x_o^2}{8}\sec^2\theta-\frac{1}{2}(1-\frac{x_o^2}{4}\sec^2\theta)\cos(x_o\sec\theta)\right.$$

$$\left.-\frac{x_o}{2}\sec\theta\sin(x_o\sec\theta)\right] \tag{2.23}$$

Substituting Equation (2.14) into Equation (2.13),

$$\frac{R\pi K_o^4}{\rho a_1^2U^2} = (\frac{x_o}{2})^2\,[P_3(0,t)-\frac{1}{\lambda^3}P_5(0,t)+P_3(-\frac{x_o}{2},t)$$

$$-\frac{4}{x_o}P_4(x_o,t)+\frac{4}{x_o^2}P_5(x_o,t)] \tag{2.24}$$

where the right hand side should be evaluated at $t = 0$, t_o, and $2t_o$ and the results added as follows:

$$f(t=0) - 2f(t=t_o) + f(t=2t_o)$$

DISCUSSION

by E. O. Tuck

I wish to make a comment which applies whenever attempts are made to obtain wave height (or equivalently, pressure) estimations near to disturbing bodies. Although Professor Takahei expressly indicates that he is using the Inui rather than the Michell approach, I take it that his ships are nevertheless thin, since for instance he calculates $\xi(x)$ on the centre plane where there B in fact no water, only ship, unless the ship is thin enough to be approximated by its centre plane. Therefore, my remarks concern calcualtion of wave profiles near or on the hull on the basis of thin or slender ship theories.

Professor Takahie's Equation (2) is presumably obtained from the usual linearized result

$$g\,\zeta = [-U\phi_x]_{z=o} \,.$$

I have doubts about whether this gives the whole story near a thin or slender ship. One must always remember that linearization is only valid as a <u>consequence</u> of thinness or slenderness; this is a warning that nonlinear effects may be important in any calculation of flow <u>near</u> the ship. Thus in the present case the exact formula is

$$g\,\zeta = [-U\phi_x - \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2)]_{z\,=\,\xi}$$

and I believe that at least the term "ϕ_y^2" can sometimes be of comparable important to the linear term "$U\phi_x$" near the ship.

For instance, in the case of a slender ship where we have $\phi = O(\mathcal{E}^2)$ and $\frac{\partial}{\partial y}, \frac{\partial}{\partial z} = O(\frac{1}{\mathcal{E}})$ near the ship c.f. $\frac{\partial}{\partial x}$, then $U\phi_x$ and ϕ_y^2 are both of $O(\epsilon^2)$. Clearly ϕ_x^2 is of order ϵ^4; the fact that ϕ_z^2 is also of order ϵ^4 follows from inspection of the exact kinematic free surface condition

$$\phi_z = U\zeta_x + \phi_x\zeta_x + \phi_y\zeta_y \,.$$

Hence the correct formula for wave height near the ship would seem to be

$$g\zeta = [-U\phi_x - \frac{1}{2}\phi_y^2]_{z=0} \,.$$

Although this formula is derived for the case of a slender ship, it may possibly also be valid for thin ships.

As an example, we might consider the case of a body of revolution $r = r_o(x)$, in which case ζ takes the form

$$g\zeta = -U\,[a'(x) \log y + b'(x)] - \frac{1}{2}\frac{(a(x))^2}{y^2}$$

where $a(x) = \frac{U}{2\pi} S'(x) = Ur_o(x)r_o'(x)$ and $b(x)$ is a certain functional of $a(x)$ defined in my paper at this seminar. On the body itself $y = r_o(x)$, so that $\zeta(x)$ is given by

$$g\zeta(x) = -U[a'(x) \log r_o(x) + b'(x)] - \frac{1}{2} U^2(r_o'(x))^2$$

I feel that some term analogous to the last term of this equation should occur in Professor Takahei's results.

AUTHOR'S REPLY

I appreciate very much your discussion to my paper. I certainly agree with you in thinking of additional steps of the procedure presented here. In this respect you have made a helpful suggestion.

As you mentioned, I started from the result of the usual linearized theory and have worked on according to the assumption that the calculated wave profile on the model central plane may be substituted for the wave profile alongside the model because of the stationary property of the wave pattern in the vicinity of the model. I made use of the same procedure previously in my wdrk with the mathematical models,[6] the measured wave profiles yielding good agreement with the calculated ones within moderate speed range (Figure 10). Later on I was engaged in wave analysis with the models of conventional hull shape, and have found less correlation between the measured wave profiles and the calculated ones. The discrepancies seem to be greater with decreasing model speed. In general a conventional model is designed with more compound curvature than are the mathematical models. As you suggested, non-linear effects may be serious in wave profile calculation near the ship, particularly in low to moderate speed range with models formed by compound curvature.

Then my work should be extended to include second order corrections. Under the second order corrections are included all the steps necessary to rectify the errors resulting from the approximations made in the first-order or linearized theory. These are the corrections for the boundary condition at the hull surface as well as for the boundary condition at the free surface. In addition viscosity effect certainly has to be taken into account.

SOME MATHEMATICAL TABLES FOR THE DETERMINATION OF WAVE PROFILES

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1. INTRODUCTION

Studies in the wave-resistance of ship hull forms have hitherto dealt primarily with the relationship between the hull form and the drag force, an integrated final result of wave-making phenomenon. The resistance test is a quite practical means to predict the necessary horsepower to propel a ship and has been used for nearly a hundred years as it had been devised by W. Froude. It is, however, less instructive as a clue in clarifying the mechanism of wave-making phenomenon and particularly in attaining prospective improvement of the ship's hull form. On the other hand, little attention has been given so far to the wave pattern or the wave profile despite the fact that these configurations could be seen advancing in beauty in a routine tank test. This may have primarily been due to the complicated experimental and theoretical procedures involved in the observation and analysis of wave patterns. With modern equipment and techniques, such as observation of wave profile using photographs, mapping of wave contours using photogrammetry or acoustic transducers, etc., the difficulties in measuring model waves can be avoided. High-speed computers are also available for the tedious theoretical calculations used in determining wave patterns. The method of wave analysis based on the measured and the calculated wave profile has now become a subject matter in the research of wave-resistance. It works substantially in checking the validity of linearized boundary conditions which the theory has employed and applying viscous corrections to the theory of the wave-resistance of an inviscid fluid. In particular, the waveless hull form associated with the bulb is to be designed to attain the most favorable interference between bow wave and bulb wave. For this purpose wave observation and its analysis play the leading role in carrying out the tests.

Generally speaking, the calculation pertaining to wave profiles is tedious and a great amount of work is involved in it. This is the reason that the simplest case of infinite draft model has been primarily dealt with so far.[1-7] The case of finite draft now ought to be taken into account by taking advantage of computers. Once the calculation of wave profile or wave pattern is carried out mechanically, the wave analysis should be utilized more often than not in investigations of wave-making resistance.

With these considerations, the Subcommittee of the Wave-making Resistance of the Japan Towing Tank Committee has undertaken the compilation of numerical tables which are available primarily for determining the wave profile alongside the ship's center plane.

The fundamental part of these tables has been completed recently. It is not only very useful in wave analysis as it stands, but also may be used further to derive associated functions.

The magnitude of this project is so great that we have conceived the idea of having an international committee organized for this purpose. The committee's duty will be at first to determine the line of approach, and then to establish the formulas of functions to be worked with, to share the work involved in making numerical tables, and finally to compile and publish the tables. We hope that our work which has been done so far in Japan will become the impetus for further development in this direction.

2. THE SCOPE OF THE TABLES MADE IN JAPAN

The tables completed so far in Japan pertain to the wave profile on the vertical plane including the direction of the approach of the advancing singularity. They are classified into two categories.

2.1 Surface Elevation Due to a Continuous Source Distribution.

The ship's wave theoretically can be calculated more accurately with a singularity distribution corresponding to the hull form than with the configuration of the hull form substituted by the Michel's approximation. The former is the one with which we worked. However, we have not as yet had an exact correlation between the hull configuration and a singularity distribution. In particular, we lack knowledge concerning the singularity distribution defining the prescribed hull form. Under these circumstances, before undertaking our work we carefully examined the following terms in conjunction with the singularity distribution.

a) The place to locate the singularity; i.e., on a vertical plane, on a horizontal plane, on a general plane, on a general curved surface, or a volumetric distribution.

b) The functional form of the distribution, which is whether the variables should be separated or not separated. The former is in the form of a product of two functions each dependent upon only one of the two space coordinates of the center plane. If the separated form is preferable, what is an appropriate functional form for each of the variables, i.e., draftwise and lengthwise respectively?

The problem is closely connected with the future approach to ship hydrodynamics. Regardless of the above cited cases, there is certainly a need for the simplest case, that of a vertical line distribution with a finite depth of uniform strength, which is the case we have worked out.

2.2 Surface Elevation Due to an Isolated Doublet.

The isolated doublet arises in shiphydrodynamics concerning the problem of a bulbous bow.[8] The bulb can be substituted by an isolated doublet or a distributed doublet advancing at a certain depth under the water.

In this connection numerical tables regarding surface elevation due to an isolated source have been made at the National Physical Laboratory, England. The surface elevation of a doublet and that of a source are related in such a way that the former is obtained by the derivative of the latter with respect to a coordinate in the direction of the doublet axis. The table concerned with the free traveling wave due to an isolated doublet has been formed by a numerical derivation of the table compiled by the National Physical Laboratory.

In addition, a table for the local disturbance due to an isolated doublet has been made.

3. DEFINITIONS OF FUNCTIONS

3.1 Nomenclature.

- F : Froude number
- L : Model length
- K_o : g/v^2
- K_oL : $gL/v^2 = 1/F^2$
- T : Depth of singularity distribution along a vertical line
- t : K_oT
- v : Speed of advance on the direction of the x-axis
- m : Strength of line singularity per unit vertical length, being equal to total flux out of singularity per unit depth as in the uniform flow of a unit speed
- x : x-axis coordinate in a unit of K_oL
- ξ : x-axis coordinate of singularity involved
- ζ : surface elevation, upward taken as positive.

The range covered by the table is presented in the following manner.

0.0 (0.1) 1.0

This means that the parameter changes from 0.0 through 1.0 with spacing of 0.1.

3.2 Calculation of Surface Elevation.

When the ship is advancing in the positive direction of x-axis with the midship at the origin, the surface elevation at the location, x, due to a singularity distribution, $m(\xi)$, is given as

$$\zeta(x)/L = 1/(\pi K_o L) \int_{-K_o L/2}^{K_o L/2} m(\xi)\, U(x-\xi,t)\, d\xi \qquad (1)$$

where $U(x-\xi,t)$ is the function which is to be evaluated by taking advantage of the tables.

The function $U(x-\xi,t)$ takes a different form depending upon whether the location at which surface elevation is evaluated is in front of a singularity or in the rear of it; that is,

$$x > \xi \;:\; U(x-\xi,t) = O^{(1)}_{-1}(x-\xi,0) - O^{(1)}_{-1}(x-\xi,t) \qquad (2)$$

$$x \leq \xi \;:\; U(x-\xi,t) = 2P_{-1}(\xi-x,0) - 2P_{-1}(\xi-x,t)$$

$$-\{O^{(1)}_{-1}(\xi-x,0) - O^{(1)}_{-1}(\xi-x,t)\} \qquad (3)$$

where $O^{(1)}_{-1}$ and P_{-1} pertain to a local surface elevation and a traveling free wave respectively. The profile of the local elevation and that of the traveling wave are giving in terms of

$$O^{(1)}_{-1}(x-\xi,0) - O^{(1)}_{-1}(x-\xi,t)$$

and

$$2\{P_{-1}(x-\xi,0) - P_{-1}(x-\xi,t)\}$$

respectively.

The functions $O^{(1)}_{-1}(u,t)$ and $P_{-1}(u,t)$ all defined as the case of $n = -1$ in the following formulas:[9]

$$P_n(u,t) = \frac{1}{2}\{O_n^{(1)}(u,t) - O_n^{(2)}(u,t)\} \tag{4}$$

$$Q_n(u,t) = \frac{1}{2}\{O_n^{(1)}(u,t) - O_n^{(2)}(u,t)\} \tag{5}$$

$$O_n^{(1)}(u,t) = \lim_{\mu\to+0} \frac{(-i)^n}{4\pi} \int_{-\pi}^{\pi} du \int_0^{\infty} \frac{e^{-Kt + iK\cos u}}{K\cos^2 u - 1 + \mu i\cos u} \cos^{n+2} u \, du \tag{6}$$

$$O_n^{(2)}(u,t) = \lim_{\mu\to+0} \frac{i^n}{4\pi} \int_{-\pi}^{\pi} du \int_0^{\infty} \frac{e^{-Kt - iK\cos u}}{K\cos^2 u - 1 + \mu i\cos u} \cos^{n+2} u \, du \tag{7}$$

where $u > 0$, $t > 0$ and n is integer. In case of $u < 0$, they are defined as follows.

$$O_n^{(1)}(u,t) = (-)^n O_n^{(1)}(-u,t) \tag{8}$$

$$O_n^{(2)}(u,t) = (-)^n O_n^{(2)}(-u,t) \tag{9}$$

The functions defined above are related to the functions $P_n(u)$ and $Q_n(u)$ introduced by T. H. Havelock in the following ways.

$$P_n(u,0) = P_n(u) \tag{10}$$

$$P_n(u,0) = -\frac{\pi}{2} \int_{\infty}^{x} \cdots \int_{\infty}^{x} Y_o(x)(dx)^{n+1} \tag{11}$$

$$P_{-n}(u,0) = -\frac{\pi}{2} \left(\frac{d}{dx}\right)^{n-1} Y_o(x) \tag{12}$$

$$O_{-1}^{(1)}(u,0) = \frac{\pi}{4}\{H_o(x) - Y_o(x)\} = Q_{-1}(u) \tag{13}$$

$$Q_n(u) = \int_0^x Q_{n-1}(t)\, dx \tag{14}$$

In connection with the formulas stated above, the numerical tables of the following functions:

1 $\quad P_{-1}(u,t)$

# 2	$O^{(1)}_{-1}(u,t)$
# 3	$P_{-1}(u,0) - P_{-1}(u,t)$
# 4	$O^{(1)}_{-1}(u,0) - O^{(1)}_{-1}(u,t)$

have been made for the following ranges of ship speed and the ratios of the depth of a singularity distribution to its length:

K_oL ; 2 through 26 (speed-length ratio = 2.38 through 0.66)
T/L ; 0.03, 0.04, 0.05

so as to be able to deal with most conventional surface ships. The tables feature in giving a component of local elevation and that of a free wave separately. The tables of $P_{-1}(u,t)$ and $O^{(1)}_{-1}(u,t)$ cover the range

t : 0 through 5.2
u : 0 through 27.0

The preliminary print of the tables takes 66 pages.

4. CALCULATION PROCEDURE OF THE FUNCTIONS

In order to evaluate the values of $P_{-1}(u,t)$, $O^{(1)}_{-1}(u,t)$, numerical integration was commonly used for cases of variables in range #2 as shown in the adjoining table. Appropriate asymptotic expansions were made use of for variables in ranges #1 and #3.

Range #1	$u^2/4t \leq 4$
Range #2	$u^2/4t \geq 4$ and $u \leq 5$
Range #3	$u \geq 5$

4.1 $P_{-1}(u,t)$ Calculation

(a) Range #1:

$$P_{-1}(u,t) = \sum_{n=0}^{\infty} \frac{(-1)^n u^{2n}}{(2n)!} U_n(t)$$

where $U_o(t) = \frac{1}{2} e^{-\frac{t}{2}} K_o(\frac{t}{2})$

$$U_1(t) = \frac{1}{4} e^{-\frac{t}{2}} \{K_1(\frac{t}{2}) + K_o(\frac{t}{2})\}$$

herein K_o and K_1 are the modified Bessel functions.

For further orders of n the following recurrence formula is applicable.

$$U_{n+1}(t) = (1 + \frac{n}{t}) U_n(t) - \frac{(n + \frac{1}{2})}{t} U_{n-1}(t)$$

(b) Range #2:

$$P_{-1}(u,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{v^2}{4t}} P_{-1}(|u + v|,0)dv$$

where $P_{-1}(u,0) = -\frac{\pi}{2} Y_o(u)$

wherein Y_o is a Bessel function of the second kind.

(c) Range #3:

The following series expansion may be used to obtain accuracy to four or five significant figures.

$$P_{-\nu-1}(u,t) \approx \sqrt{\frac{\pi}{2r}} e^{-t} \{C \cos Cu + \frac{\nu}{2} \pi) - S \sin (u + \frac{\nu}{2} \pi)\}$$

where $C = \cos(\frac{\theta}{2}) - \frac{3a}{4r^2} \cos(\frac{5}{2} \theta) - \frac{15b}{4r^3} \cos(\frac{7}{2} \theta)$

$$+ \frac{3}{16r^4} (a^2 - 35c) \cos(\frac{9}{2} \theta) - \cdots$$

$$S = \sin(\frac{\theta}{2}) - \frac{3a}{4r^2} \sin(\frac{5}{2} \theta) - \frac{15b}{4r^2} \sin(\frac{7}{2} \theta)$$

$$+ \frac{3}{16r^4} (a^2 - 35c) \cos(\frac{9}{2} \theta) - \cdots$$

$$r = \sqrt{u^2 + (2t + \frac{I}{4} - \nu)^2}$$

$$\tan\theta = u/(2t - \nu + \frac{1}{4})$$

$$a = t + \frac{1}{2}\left(\nu - \frac{1}{8}\right)$$

$$b = -\frac{1}{3}\left(\nu - \frac{1}{16}\right)$$

$$c = \frac{1}{4}\left(\nu - \frac{1}{32}\right)$$

and $$P_{-1}(0,t) = U_o(t)$$

4.2 $O_{-1}^{(1)}(u,t)$ Calculation

(a) Range #1 and #2:

$$O_{-1}^{(1)}(u,t) = R_{-1}(u,t) + S_{-1}(u,t) + P_{-1}(u,t)$$

where

$$R_{-1}(u,t) = -\frac{1}{2}\int_{-1}^{1} \frac{e^{-\frac{u^2}{4t}(1-v^2)}}{\sqrt{\left((1+v)^2 + \frac{4t^2}{u^2}\right)}}\, dv$$

$$S_{-1}(u,t) = \int_0^{\pi/2} e^{-t\cos^2\theta} \sin(u\cos\theta)\, d\theta$$

(b) Range #3:

$$O \quad (u,t) \approx a + b + c + d + e$$

where
$$a = \frac{1}{2u}(1-y)$$

$$b = \frac{1}{u^2 z}\left(1 - \frac{3}{2}y + \frac{1}{2}y^3\right)$$

$$c = \frac{1}{u^3}\left\{-\frac{1}{2} + \frac{1}{z^2}\left(6 - 10y + \frac{11}{2}y3 - \frac{3}{2}y^5\right)\right\}$$

$$d = \frac{1}{u^4 z}\left\{-6 + \frac{1}{z^2}\left(60 - 105y + \frac{147}{2}y^3 - 36y^5 + \frac{15}{2}y^7\right)\right\}$$

$$e = \frac{1}{u^5}\left\{-\frac{9}{2} + \frac{90}{z^2} - \frac{1}{z^4}\left(840 - 1512y + 1224y^3 - 814.5y^5 + 315y^7 - 52.5y^9\right)\right\}$$

$$z = \frac{u}{t}, \quad y = 1/\sqrt{1+z^2}$$

and also

$$O_{-1}^{(1)}(u,0) = \frac{\pi}{4}\left\{H_o(u) - Y_o(u)\right\}$$

$$O_{-1}^{(1)}(0,t) = P_{-1}(0,t)$$

herein H_o is a Struve's function.

5. TABLES OF WAVE PROFILE DUE TO POLYNOMIAL SINGULARITY DISTRIBUTIONS

Effective use of the tables, before their completion, has been made in several works on wave analysis.[10,11]

As in most of the cases, the distribution function of a singularity is represented by a polynominal in a coordinate along the ship's length. The tables of $P_{-1}(u,t)$ and $O_{-1}^{(1)}(u,t)$ have been compiled so as to make it possible to determine a wave profile due to a polynomial singularity distribution. Surface elevation at a appropriate number of stations is tabulated as a contribution from each term of a polynomial.

Tables 1 through 6 are for the case in which the ratio of the depth of the distribution to its length is 0.04. An example of use of this table may be of interest. A singularity distribution being given:

$$m(\xi) = 1.63213\xi \pm 1.13213\xi^4$$
$$- \text{ for } 0 \leq \xi \leq 1$$
$$+ \text{ for } 1 \leq \xi \leq 0$$

the wave profile at the velocity, K_oL of 16 or a Froude number of 0.25, is obtained very readily by taking advantage of Table 3. All that needs to be done is to sum up the values in the table multiplied by values of the coefficients of the polynomial. Figure 1 is the obtained results, consisting of a local elevation, a free traveling wave and a total wave.

6. WAVE PROFILE DUE TO A DOUBLET

The surface elevation, ζ_D, at $P(x,0,0)$ generated by a unit doublet directed toward the positive x-axis located at $Q(x',0,t')$ and associated with a uniform flow of velocity, v, coming in the direction of the negative x-axis is our concern. Herein the coordinates of P and Q are in units of K_o and the strength of the doublet, M, is related to a sphere of radius, a, placed in an infinite fluid of velocity, v, in such a way that $M = 2\pi a_o^3 v$. In front of the doublet $(x > x')$, only the local elevation exists and

$$\zeta_D(P,Q) = \frac{K_o^2}{\pi v} Z_{-4}(x - x',t)$$

where, assuming $u > 0$,

TABLE 1(a)

ζ_l , ζ_w , ζ At $K_oL = 25$

		T/L = 0.04					K₀L = 25 (F = 0.2000)						
u	$\frac{x}{L/2}$	n = 1			n = 2			n = 3			n = 4		
		$\zeta_l^{(1)}/L \times 100$	$\zeta_w^{(1)}/L \times 100$	$\zeta^{(1)}/L \times 100$	$\zeta_l^{(2)}/L \times 100$	$\zeta_w^{(2)}/L \times 100$	$\zeta^{(2)}/L \times 100$	$\zeta_l^{(3)}/L \times 100$	$\zeta_w^{(3)}/L \times 100$	$\zeta^{(3)}/L \times 100$	$\zeta_l^{(4)}/L \times 100$	$\zeta_w^{(4)}/L \times 100$	$\zeta^{(4)}/L \times 100$
13.5	1.080	.2284		.2284	.1989		.1989	.1778		.1778	.1617		.1617
13.0	1.040	.3557		.3557	.3174		.3174	.2891		.2891	.2670		.2670
12.5	1.000	.7962	0	.7962	.7406	0	.7406	.6977	0	.6977	.6627	0	.6627
12.0	.960	.3071	1.8561	2.1631	.2364	1.8064	2.0428	.1822	1.7584	1.9405	.1386	1.7118	1.8505
11.5	.920	.1505	1.8527	2.0032	.0758	1.7295	1.8053	.0218	1.6146	1.6364	-.0188	1.5075	1.5056
10.5	.840	.0112	.4217	.4329	-.0587	.2177	.1590	-.1003	.0480	- .0523	-.1249	- .0926	- .2174
9.5	.760	-.0561	- .9160	- .9721	-.1126	-1.0749	-1.1875	-.1364	-1.1737	-1.3102	-.1431	-1.2284	-1.3715
8.5	.680	-.0966	-1.2017	-1.2983	-.1355	-1.2520	-1.3875	-.1410	-1.2331	-1.3741	-.1326	-1.1743	-1.3069
7.5	.600	-.1241	- .5546	- .6787	-.1428	- .5203	- .6631	-.1317	- .4264	- .5581	-.1128	- .3130	- .4257
6.5	.520	-.1447	.2933	.1486	-.1417	.3440	.2023	-.1170	.4353	.3183	-.0920	.5222	.4302
5.5	.440	-.1593	.6178	.4584	-.1340	.6368	.5028	-.0994	.6780	.5785	-.0724	.6989	.6265
4.5	.360	-.1700	.2310	.0610	-.1225	.2264	.1040	-.0820	.2288	.1468	-.0561	.2056	.1495
3.5	.280	-.1776	- .4454	- .6231	-.1088	- .4244	- .5332	-.0664	- .4141	- .4805	-.0438	- .4294	- .4732
2.5	.200	-.1830	- .7991	- .9821	-.0946	- .7092	- .8038	-.0539	- .6583	- .7122	-.0354	- .6339	- .6692
1.5	.120	-.1864	- .5585	- .7449	-.0814	- .3990	- .4803	-.0453	- .3205	- .3658	-.0304	- .2700	- .3004
0.5	.040	-.1880	.0116	- .1764	-.0716	.2026	.1309	-.0408	.2604	.2195	-.0282	.2955	.2674

TABLE 1(b)

ζ_l , ζ_w , ζ At $K_oL = 25$

		T/L = 0.04					K₀L = 25 (F = 0.2000)						
u	$\frac{x}{L/2}$	n = 1			n = 2			n = 3			n = 4		
		$\zeta_l^{(1)}/L \times 100$	$\zeta_w^{(1)}/L \times 100$	$\zeta^{(1)}/L \times 100$	$\zeta_l^{(2)}/L \times 100$	$\zeta_w^{(2)}/L \times 100$	$\zeta^{(2)}/L \times 100$	$\zeta_l^{(3)}/L \times 100$	$\zeta_w^{(3)}/L \times 100$	$\zeta^{(3)}/L \times 100$	$\zeta_l^{(4)}/L \times 100$	$\zeta_w^{(4)}/L \times 100$	$\zeta^{(4)}/L \times 100$
- 0.5	- .040	-.1880	.3847	.1967	-.0716	.5637	.4921	-.0408	.5606	.5198	-.0282	.5530	.5248
- 1.5	- .120	-.1864	.2490	.0626	-.0814	.3705	.2890	-.0453	.3388	.2935	-.0304	.3017	.2713
- 2.5	- .200	-.1830	- .2388	- .4218	-.0946	- .1799	- .2745	-.0539	- .1770	- .2309	-.0354	- .2010	- .2364
- 3.5	- .280	-.1776	- .6232	- .8009	-.1088	- .5881	- .6970	-.0664	- .5218	- .5882	-.0438	- .4985	- .5423
- 4.5	- .360	-.1700	- .5698	- .7397	-.1225	- .5238	- .6462	-.0820	- .4175	- .4994	-.0561	- .3497	- .4058
- 5.5	- .440	-.1593	- .1521	- .3114	-.1340	- .0986	- .2326	-.0994	- .0059	- .1053	-.0724	.0711	- .0013
- 6.5	- .520	-.1447	.2374	.0928	-.1417	.2637	.1220	-.1170	.3015	.1845	-.0920	.3507	.2587
- 7.5	- .600	-.1241	.2527	.1286	-.1428	.2207	.0779	-.1317	.2032	.0714	-.1128	.2145	.1017
- 8.5	- .680	-.0966	- .1010	- .1976	-.1355	- .1897	- .3252	-.1410	- .2307	- .3717	-.1326	- .2388	- .3714
- 9.5	- .760	-.0561	- .4889	- .5450	-.1126	- .6041	- .7167	-.1364	- .6392	- .7757	-.1431	- .6455	- .7885
-10.5	- .840	.0112	- .5617	- .5506	-.0587	- .6734	- .7321	-.1003	- .7067	- .8070	-.1249	- .7156	- .8405
-11.5	- .920	.1505	- .2677	- .1172	.0758	- .3740	- .2983	.0218	- .4424	- .4207	.0188	- .4903	- .4715
-12.0	- .960	.3071	- .0679	.2392	.2364	- .1802	.0562	.1822	- .2825	- .1003	.1386	- .3677	- .2291
-12.5	-1.000	.7962	.1135	.9096	.7406	- .0137	.7269	.6977	- .1582	.5395	.6627	- .2921	.3706

ζ_l , ζ_w , ζ AT $K_oL = 25$

TABLE 2

ζ_ℓ , ζ_w , ζ At $K_oL = 20$

		T/L = 0.04			K.L = 20 (F = 0.2236)								
u	$\frac{x}{L/2}$	n = 1			n = 2			n = 3			n = 4		
		$\zeta_\ell^{(1)}/L\times100$	$\zeta_w^{(1)}/L\times100$	$\zeta^{(1)}/L\times100$	$\zeta_\ell^{(2)}/L\times100$	$\zeta_w^{(2)}/L\times100$	$\zeta^{(2)}/L\times100$	$\zeta_\ell^{(3)}/L\times100$	$\zeta_w^{(3)}/L\times100$	$\zeta^{(3)}/L\times100$	$\zeta_\ell^{(4)}/L\times100$	$\zeta_w^{(4)}/L\times100$	$\zeta^{(4)}/L\times100$
11.0	1.1000	.2106		.2106	.1806		.1806	.1596		.1596	.1439		.1439
10.5	1.0500	.3362		.3362	.2963		.2963	.2676		.2676	.2454		.2454
10.0	1.0000	.8261	0	.8261	.7655	0	.7655	.7194	0	.7194	.6820	0	.6820
9.5	.9500	.2750	2.1696	2.4446	.1971	2.0961	2.2932	.1387	2.0255	2.1641	.0926	1.9576	2.0503
9.0	.9000	.1139	2.0672	2.1811	.0336	1.8901	1.9237	-.0218	1.7281	1.7064	-.0613	1.5799	1.5186
8.0	.8000	-.0269	.3392	.3123	-.0956	.0640	- .0316	-.1309	- .1518	- .2826	-.1473	- .3197	- .4670
7.0	.7000	-.0950	-1.1114	-1.2064	-.1415	-1.3042	-1.4458	-.1521	-1.3993	-1.5514	-.1467	-1.4295	-1.5762
6.0	.6000	-.1359	-1.[illegible]	-1.4999	-.1552	-1.3939	-1.5491	-.1430	-1.3212	-1.4642	-.1226	-1.2024	-1.3249
5.0	.5000	-.1634	- .6273	- .7907	-.1530	- .5341	- .6870	-.1230	- .3681	- .4911	-.0950	- .1999	- .2949
4.0	.4000	-.1838	.2918	.1080	-.1428	.4133	.2704	-.1012	.5592	.4579	-.0721	.6646	.5925
3.0	.3000	-.1973	.6209	.4236	-.1266	.7076	.5810	-.0801	.7693	.6892	-.0540	.7694	.7154
2.0	.2000	-.2059	.1792	- .0267	-.1082	.2462	.1380	-.0631	.2410	.1779	-.0421	.1792	.1371
1.0	.1000	-.2107	- .5586	- .7694	-.0914	- .4425	- .5339	-.0522	- .4538	- .5060	-.0358	- .4992	- .5350
0	0	-.2123	- .9285	-1.1408	-.0823	- .7056	- .7879	-.0485	- .6875	- .7359	-.0339	- .6713	- .7052
-1.0	- .1000	-.2107	- .6498	- .8605	-.0914	- .3552	- .4466	-.0522	- .3037	- .3559	-.0358	- .2412	- .2770
-2.0	- .2000	-.2059	- .0252	- .2311	-.1082	.2062	.0979	-.0631	.2984	.2353	-.0421	.3588	.3167
-3.0	- .3000	-.1973	.3705	.1732	-.1266	.4563	.3297	-.0801	.5556	.4755	-.0540	.5814	.5274
-4.0	- .4000	-.1838	.2083	.0245	-.1428	.1687	.0258	-.1012	.2391	.1378	-.0721	.2426	.1706
-5.0	- .5000	-.1634	- .3281	- .4915	-.1530	- .4087	- .5616	-.1230	- .3730	- .4960	-.0950	- .3510	- .4460
-6.0	- .6000	-.1359	- .7391	- .8750	-.1552	- .7911	- .9463	-.1430	- .7748	- .9178	-.1226	- .7103	- .8329
-7.0	- .7000	-.0950	- .6682	- .7632	-.1415	- .6936	- .8352	-.1521	- .6911	- .8432	-.1467	- .6096	- .7563
-8.0	- .8000	-.0269	- .2071	- .2340	-.0956	- .2701	- .3657	-.1309	- .3050	- .4359	-.1473	- .2731	- .4204
-9.0	- .9000	.1139	.2119	.3258	.0336	.0455	.0791	-.0218	- .0691	- .0908	-.0613	- .1553	- .2167
-9.5	- .9500	.2750	.2734	.5484	.1971	.0465	.2436	.1387	- .1227	.0160	.0926	- .2848	- .1921
-10.0	-1.0000	.8261	.2172	1.0433	.7655	- .0622	.7034	.7194	- .2894	.4300	.6820	- .5299	.1520

TABLE 3

ζ_ℓ , ζ_w , ζ At $K_oL = 16$

		T/L = 0.04			K.L = 16 (F = 0.2500)								
u	$\frac{x}{L/2}$	n = 1			n = 2			n = 3			n = 4		
		$\zeta_\ell^{(1)}/L\times100$	$\zeta_w^{(1)}/L\times100$	$\zeta^{(1)}/L\times100$	$\zeta_\ell^{(2)}/L\times100$	$\zeta_w^{(2)}/L\times100$	$\zeta^{(2)}/L\times100$	$\zeta_\ell^{(3)}/L\times100$	$\zeta_w^{(3)}/L\times100$	$\zeta^{(3)}/L\times100$	$\zeta_\ell^{(4)}/L\times100$	$\zeta_w^{(4)}/L\times100$	$\zeta^{(4)}/L\times100$
9.0	1.1250	.1908		.1908	.1610		.1610	.1407		.1407	.1258		.1258
8.5	1.0625	.3112		.3112	.2707		.2707	.2422		.2422	.2205		.2205
8.0	1.0000	.8472	0	.8472	.7825	0	.7825	.7337	0	.7337	.6944		.6944
7.5	.9375	.2353	2.5060	2.7413	.1514	2.3982	2.5497	.0902	2.2958	2.3859	.0431	2.1984	2.2415
7.0	.8750	.0724	2.2520	2.3244	-.0109	2.0016	1.9908	-.0648	1.7782	1.7135	-.1007	1.5789	1.4782
6.0	.7500	-.0693	.1995	.1301	-.1306	- .1591	- .2897	-.1546	- .4179	- .5725	-.1597	- .6014	- .7611
5.0	.6250	-.1379	-1.3325	-1.4704	-.1648	-1.5458	-1.7106	-.1571	-1.6083	-1.7653	-.1386	-1.5847	-1.7233
4.0	.5000	-.1786	-1.5375	-1.7161	-.1657	-1.5077	-1.6734	-.1332	-1.3425	-1.4757	-.1032	-1.1428	-1.2461
3.0	.3750	-.2050	- .7134	- .9184	-.1513	- .5036	- .6549	-.1044	- .2397	- .3441	-.0734	- .0236	- .0970
2.0	.2500	-.2236	.2671	.0435	-.1314	.5266	.3953	-.0807	.7230	.6423	-.0548	.8119	- .7571
1.0	.1250	-.2334	.5915	.3581	-.1093	.8202	.7109	-.0641	.8612	.7972	-.0444	.7959	.7515
0	0	-.2365	.0931	- .1434	-.0962	.3172	.2210	-.0582	.2263	.1681	-.0413	.0992	.0578
-1.0	- .1250	-.2334	- .7003	- .9337	-.1093	- .4460	- .5552	-.0641	- .5351	- .5992	-.0444	- .5936	- .6380
-2.0	- .2500	-.2236	-1.0787	-1.3032	-.1314	- .8612	- .9926	-.0807	- .7772	- .8579	-.0548	- .6996	- .7544
-3.0	- .3750	-.2050	- .7594	- .9644	-.1513	- .6313	- .7826	-.1044	- .3896	- .4940	-.0734	- .2178	- .2912
-4.0	- .5000	-.1786	- .0847	- .2633	-.1657	- .0738	- .2395	-.1332	.1632	.0300	-.1032	.3238	.2206
-5.0	- .6250	-.1379	.3264	.1885	-.1648	.2185	.0538	-.1571	.2907	.1336	-.1386	.3561	.2175
-6.0	- .7500	-.0693	.3849	.3156	-.1306	.1753	.0446	-.1546	.0306	- .1241	-.1597	- .0231	- .1829
-7.0	- .8750	.0724	.8437	.9161	-.0109	.4859	.4750	-.0648	.1213	.0565	-.1007	- .0962	- .1969
-7.5	- .9375	.2353	.7999	1.0315	.1514	.3284	.4799	.0902	- .1439	- .0538	.0431	- .4698	- .4267
-8.0	-1.0000	.8472	.4246	1.2718	.7825	- .1273	.6552	.7337	- .6626	.0711	.6944	-1.0756	.3812

$$Z_{-4}(u,t) = O_{-1}^{(1)}(u,t) + \frac{t}{2(u^2 + t^2)^{3/2}} - \frac{(u/t)^2}{2(u^2 + t^2)^{1/2}}$$

In the rear of the doublet $(x < x')$, the traveling wave accompanies the local elevation.

$$\zeta_D(P,Q) = \frac{K_o^2}{\pi V} \{Z_{-4}(x' - x,t) - 2P_{-4}(x' - x,t)\}$$

The latter term is the free traveling wave component.

$P_{-4}(u,t)$ has been formed by a numerical derivation of $P_{-2}(u,t)$ which had been obtained at the Mathematics Division, National Physical Laboratory (Reference No. Ma/16/1502). Because of this, the accuracy of $P_{-4}(u,t)$ is less (2 through 4 significant figures) than would be the case if it were worked with the original formulas of $P_{-4}(u,t)$. The range covered is

$\sqrt{t}$:	0.2	(0.05)	1.0
u :	0	(0.1)	60.0

For $O_{-4}^{(1)}(u,t)$ and $Z_{-4}(u,t)$, the range covered is;

u/t :	0	(0.1)	0.4	(0.2)	1.0	(0.4)	2.4
t :	0.4	(0.1)	0.8	(0.2)	1.0		

These are shown in Tables 7 and 8. These tables make it possible to determine the wave profile due to a distributed doublet along a vertical line. An example is shown in Figure 2 and 3.

TABLE 4

ζ_ℓ , ζ_w , ζ At $K_oL = 14$

T/L = 0.04		K.L = 14 (P = 0.2673)											
u	$\frac{x}{L/2}$	n = 1			n = 2			n = 3			n = 4		
		$\zeta_\ell^{(1)}/L\times100$	$\zeta_w^{(1)}/L\times100$	$\zeta^{(1)}/L\times100$	$\zeta_\ell^{(2)}/L\times100$	$\zeta_w^{(2)}/L\times100$	$\zeta^{(2)}/L\times100$	$\zeta_\ell^{(3)}/L\times100$	$\zeta_w^{(3)}/L\times100$	$\zeta^{(3)}/L\times100$	$\zeta_\ell^{(4)}/L\times100$	$\zeta_w^{(4)}/L\times100$	$\zeta^{(4)}/L\times100$
8.0	1.1429	.1808		.1808	.1504		.1504	.1302		.1302	.1156		.1156
7.5	1.0714	.2964		.2964	.2550		.2550	.2264		.2264	.2049		.2049
7.0	1.0000	.8570	0	.8570	.7892	0	.7892	.7388	0	.7388	.6985	0	.6985
6.5	.9286	.2100	2.7148	2.9248	.1226	2.5800	2.7026	.0602	2.4528	2.5129	.0132	2.3327	2.3459
6.0	.8571	.0466	2.3396	2.3862	-.0374	2.0343	1.9969	-.0890	1.7669	1.6779	-.1213	1.5327	1.4114
5.0	.7143	-.0961	.0845	- .0116	-.1496	- .3270	- .4766	-.1644	- .6047	- .7691	-.1615	- .7865	- .9480
4.0	.5714	-.1654	-1.4798	-1.6452	-.1746	-1.6902	-1.8648	-.1543	-1.7116	-1.8659	-.1285	-1.6391	-1.7676
3.0	.4286	-.2070	-1.6488	-1.8558	-.1676	-1.5501	-1.7177	-.1235	-1.3081	-1.4316	-.0905	-1.0564	-1.1469
2.0	.2857	-.2353	- .7732	-1.0085	-.1485	- .4479	- .5964	-.0949	- .1215	- .2164	-.0655	.1055	.0401
1.0	.1429	-.2511	.2392	- .0119	-.1240	.6344	.5103	-.0743	.8349	.7606	-.0519	.8834	.8315
0	0	-.2559	.5557	.2998	-.1082	.9285	.8203	-.0666	.8957	.8291	-.0477	.7717	.7240
-1.0	- .1429	-.2511	.0218	- .2292	-.1240	.3217	.1977	-.0743	.1673	.0930	-.0519	.0215	- .0303
-2.0	- .2857	-.2353	- .8023	-1.0376	-.1485	- .6576	- .8060	-.0949	- .6591	- .7540	-.0655	- .6591	- .7245
-3.0	- .4286	-.2070	-1.1820	-1.3830	-.1676	-1.1640	-1.3316	-.1235	- .9377	-1.0612	-.0905	- .7508	- .8413
-4.0	- .5714	-.1654	- .8372	-1.0026	-.1746	- .8929	-1.0675	-.1543	- .5970	- .7513	-.1285	- .3389	- .4674
-5.0	- .7143	-.0961	- .1349	- .2310	-.1496	- .2614	- .4110	-.1644	- .1422	- .3066	-.1615	.0026	- .1589
-6.0	- .8571	.0466	.2815	.3281	-.0374	.0401	.0027	-.0890	- .1724	- .2614	-.1213	- .2822	- .4035
-6.5	- .9286	.2100	.2505	.4605	.1226	- .0678	.0548	.0602	- .4518	- .3917	.0132	- .7164	- .7032
-7.0	-1.0000	.8570	.0691	.9261	.7892	- .3274	.4619	.7388	- .8581	- .1193	.6985	-1.2819	- .5834

TABLE 5

ζ_ℓ , ζ_w , ζ At $K_oL = 13$

T/L = 0.04		K.L = 13 (P = 0.2774)											
u	$\frac{x}{L/2}$	n = 1			n = 2			n = 3			n = 4		
		$\zeta_\ell^{(1)}/L\times100$	$\zeta_w^{(1)}/L\times100$	$\zeta^{(1)}/L\times100$	$\zeta_\ell^{(2)}/L\times100$	$\zeta_w^{(2)}/L\times100$	$\zeta^{(2)}/L\times100$	$\zeta_\ell^{(3)}/L\times100$	$\zeta_w^{(3)}/L\times100$	$\zeta^{(3)}/L\times100$	$\zeta_\ell^{(4)}/L\times100$	$\zeta_w^{(4)}/L\times100$	$\zeta^{(4)}/L\times100$
7.5	1.1538	.1697		.1697	.1419		.1419	.1231		.1231	.1093		.1093
7.0	1.0769	.2832		.2832	.2441		.2441	.2166		.2166	.1958		.1958
6.5	1.0000	.8575	0	.8575	.7907	0	.7907	.7403	0	.7403	.6998	0	.6998
6.0	.9231	.1923	2.8288	3.0211	.1052	2.6764	2.7816	.0430	2.5332	2.5762	-.0034	2.3988	2.3954
5.5	.8462	.0298	2.3754	2.4052	-.0525	2.0358	1.9835	-.1019	1.7419	1.6400	-.1316	1.4875	1.3559
4.5	.6923	-.1117	.0072	- .1045	-.1592	- .4329	- .5922	-.1681	- .7167	- .8848	-.1605	- .8920	-1.0525
3.5	.5385	-.1793	-1.5795	-1.7588	-.1781	-1.7786	-1.9567	-.1507	-1.7671	-1.9179	-.1216	-1.6611	-1.7827
2.5	.3846	-.2175	-1.7255	-1.9430	-.1637	-1.5700	-1.7337	-.1149	-1.2771	-1.3920	-.0818	- .9970	-1.0788
1.5	.2308	-.2389	- .8192	-1.0580	-.1359	- .4051	- .5410	-.0829	- .0462	- .1291	-.0565	.1777	.1212
0.5	.0769	-.2484	.2117	- .0367	-.1089	.7103	.6014	-.0648	.8941	.8293	-.0458	.9101	.8644
-0.5	- .0769	-.2484	.5245	.2761	-.1089	.9879	.8790	-.0648	.8951	.8303	-.0458	.7425	.6968
-1.5	- .2308	-.2389	- .0272	- .2660	-.1359	.2269	.0910	-.0829	.09[illegible]	.0157	-.0565	- .0306	- .0871
-2.5	- .3846	-.2175	- .8651	-1.0826	-.1637	- .8563	-1.0200	-.1149	- .7800	- .8948	-.0818	- .7160	- .7978
-3.5	- .5385	-.1793	-1.2418	-1.4211	-.1781	-1.3665	-1.5446	-.1507	-1.0989	-1.2497	-.1216	- .8458	- .9674
-4.5	- .6923	-.1117	- .8798	- .9915	-.1592	-1.0440	-1.2032	-.1681	[illegible]	- .9801	-.1605	- .5528	- .7133
-5.5	- .8498	.0298	- .1605	- .1308	-.0525	- .378[illegible]	- .4307	-.1019	- .4470	- .5489	-.1316	- .4288	- .5604
-6.0	- .9231	.1923	.1145	.3068	.1052	- .1635	- .0583	.0430	- .4445	- .4015	-.0034	- .6236	- .6270
-6.5	-1.0000	.8575	.2598	1.1173	.7907	- .1029	.6878	.7403	- .6042	.1361	.6998	-1.0160	- .3162

TABLE 6

ζ_L , ζ_w , ζ At $K.L = 11$

T/L = 0.04		K·L = 11 (F = 0.3015)											
u	$\frac{x}{L/2}$	n = 1			n = 2			n = 3			n = 4		
		$\zeta_L^{(1)}/L \times 100$	$\zeta_w^{(1)}/L \times 100$	$\zeta^{(1)}/L \times 100$	$\zeta_L^{(2)}/L \times 100$	$\zeta_w^{(2)}/L \times 100$	$\zeta^{(2)}/L \times 100$	$\zeta_L^{(3)}/L \times 100$	$\zeta_w^{(3)}/L \times 100$	$\zeta^{(3)}/L \times 100$	$\zeta_L^{(4)}/L \times 100$	$\zeta_w^{(4)}/L \times 100$	$\zeta^{(4)}/L \times 100$
6.5	1.1818	.1593		.1593	.1297		.1297	.1107		.1107	.0973		.0973
6.0	1.0909	.2644		.2644	.2237		.2237	.1963		.1963	.1762		.1762
5.5	1.0000	.8633	0	.8633	.7929	0	.7929	.7409	0	.7409	.6996	0	.6996
5.0	.9091	.1571	3.0940	3.2510	.0671	2.8940	2.9611	.0053	2.7085	2.7138	-.0394	2.5364	2.4971
4.5	.8182	-.0060	2.4321	2.4260	-.0852	2.0037	1.9185	-.1284	1.6443	1.5159	-.1510	1.3428	1.1917
3.5	.6364	-.1499	- .1919	- .3418	-.1787	- .6925	- .8712	-.1721	- .9748	-1.1469	-.1536	-1.1192	-1.2729
2.5	.4545	-.2190	-1.7846	-2.0037	-.1839	-1.9299	-2.1138	-.1398	-1.8212	-1.9610	-.1050	-1.6315	-1.7365
1.5	.2727	-.2598	-1.8810	-2.1408	-.1616	-1.5475	-1.7091	-.1037	-1.1369	-1.2406	-.0723	- .8140	- .8862
0.5	.0909	-.2810	- .9111	-1.1921	-.1343	- .2306	- .3649	-.0837	.1570	.0733	-.0607	.3349	.2742
-0.5	- .0909	-.2810	.1513	- .1297	-.1343	.9302	.7959	-.0837	.9961	.9124	-.0607	.9279	.8672
-1.5	- .2727	-.2598	.4446	.1848	-.1616	.8751	.7135	-.1037	.7873	.6837	-.0722	.6344	.5622
-2.5	- .4545	-.2190	- .1556	- .3746	-.1839	- .2488	- .4327	-.1398	- .2366	- .3764	-.1050	- .2337	- .3387
-3.5	- .6364	-.1499	-1.0289	-1.1788	-.1787	-1.4538	-1.6325	-.1721	-1.3094	-1.4815	-.1536	-1.0864	-1.2400
-4.5	- .8182	-.0060	-1.4015	-1.4075	-.0852	-1.8852	-1.9704	-.1284	-1.8132	-1.9416	-.1510	-1.5930	-1.7440
-5.0	- .9091	.1571	-1.2856	-1.1285	.0671	-1.7432	-1.6761	.0053	-1.8251	-1.8199	-.0394	-1.7597	-1.7991
-5.5	-1.0000	.8633	-1.0054	- .1421	.7929	-1.4498	- .6569	.7409	-1.7717	-1.0308	.6996	-1.9870	-1.2874

TABLE 7

$$O_{-4}^{(1)}(u,t)$$

t / u/t	1.0	0.8	0.9	0.6	0.5	0.4
0		-.3571	-.3854	-.4161	-.4496	-.4859
0.1	-.2419	-.2756		-.3099	-.3249	-.3346
0.2		-.1936	-.2003	-.2036	-.2000	-.1835
0.3	-.1091	-.1116		-.0979	-.0766	-.0337
0.4		-.0330	-.0192	.0045	.0431	.1122
0.6	.0867	.1284		.2113	.2859	.4066
0.8		.2628	.3366	.4103	.5402	.6918
1.0	.3331	.4306		.6029	.7471	.9705
1.6	.6771	.8551		1.1585	1.4056	1.7818
2.4	1.1101	1.3927		1.8679	2.2512	2.8300
3.2					3.0792	3.8601

TABLE 8

$$Z_{-4}(u,t)$$

t / u/t	1.0	0.8	0.7	0.6	0.5	0.4
0		.424	.635	.973	1.550	2.639
0.1	.246	.488		1.050	1.636	2.732
0.2		.518	.734	1.073	1.647	2.714
0.3	.287	.521		1.051	1.595	2.605
0.4		.499	.692	.995	1.495	2.426
0.6	.248	.428		.830	1.238	1.991
0.8		.322	.466	.655	.993	1.555
1.0	.156	.265		.505	.747	1.191
1.6	.073	.123		.235	.347	.551
2.4	.031	.053		.101	.150	.239
3.2					.078	.125

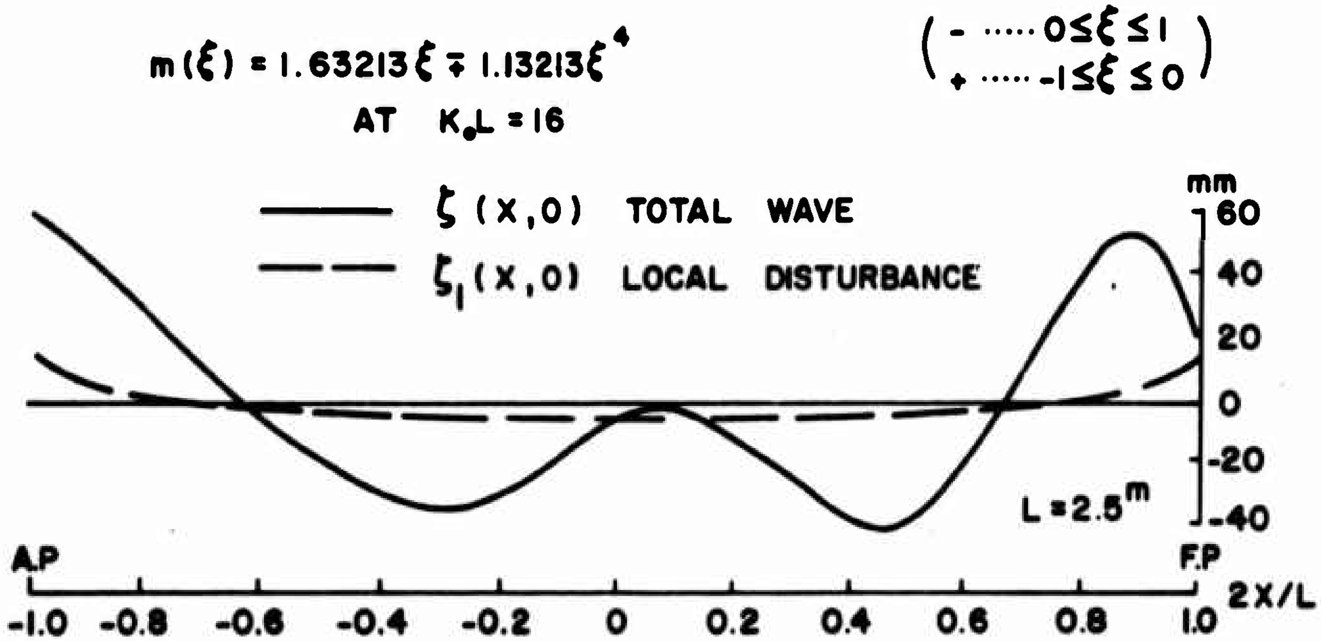

Figure 1. Wave Profile.

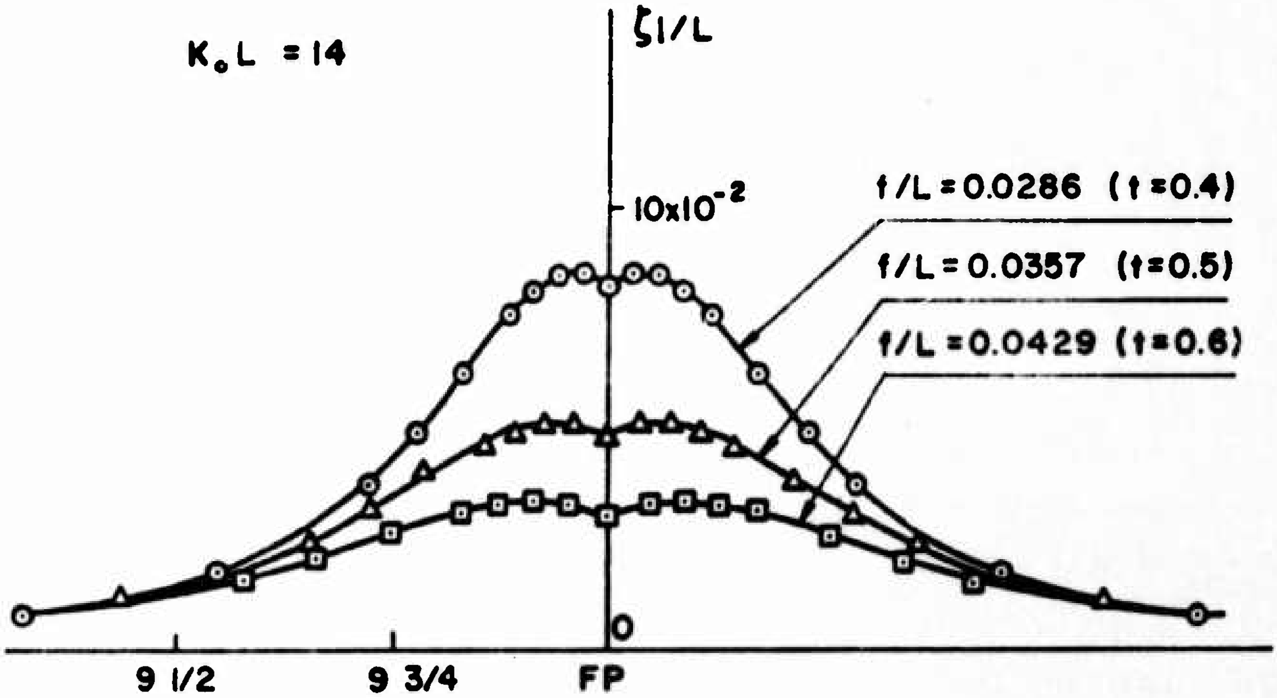

Figure 2. Local Elevation (ζ_l) Due to an Isolated Doublet or a Sphere.

a_o = radius of sphere, L = ship length
f = depth of center of sphere
K_oL = $1/(\text{Froude number})^2$, $t = K_of$.

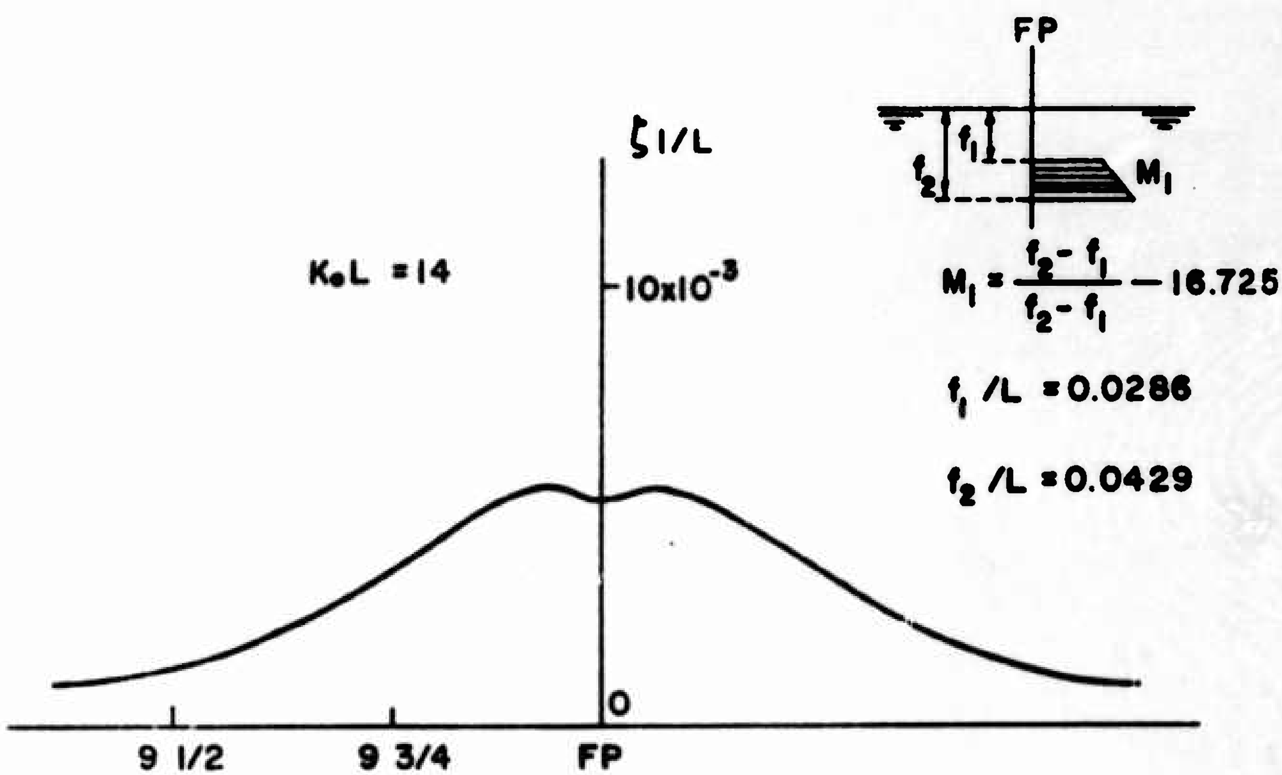

Figure 3. Local Elevation (ζ_l) Due to a Distributed Doublet.
M_1 = total strength of doublet.

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A GUIDE NOTE FOR DESIGN OF SHIP MODEL BASINS WITH SPECIAL REFERENCES TO "WAVE ANALYSIS" WORK

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A GUIDE NOTE FOR DESIGN OF SHIP MODEL BASINS
WITH SPECIAL REFERENCES TO "WAVE ANALYSIS" WORK

Abstract

With respect to the recent development in studies of wave-resistance and especially the "wave analysis" metnod, some problems involving the test facilities themselves may arise. This paper will attempt to show in what respects the existing towing tanks are not well suited to the new methods of analysis, what may be done to solve these problems while still employing the present facilities, and what important considerations must be given, should an opportunity arise to build a completely new towing tank. This paper summarizes the authors' thoughts to these questions.

The contents of the paper comprise the experience obtained from a piecemeal solution to these problems during the last ten years in The Experimental Tank of The University of Tokyo within the regular research budget (hardly sufficient) and from some major additions to The Experimental Tank facilities which have been under construction since 1962 with a special fund of some $40,000 provided by the Tōyō Rayon Foundation for the Promotion of Science and Technology and The Ministry of Education of the Japanese Government.

A. Ship Model Towing Tank and Wave-Making Resistance Theory

In scientific and technological fields, a close and complementary cooperation between theory and experiment should exist. The underlying reason is clear. The theory itself, be what it may, is always incomplete compared to a natural phenomenon with its deep inner workings of complexity and mystery. The fate of a theory, therefore, lies with the experimental verification. The best "teacher" must always be nature.

On the other hand, what a waste there would be in an experiment without a theoretical prediction.

Now, let us return to our present topic, "wave-resistance of ships." In this case, wave-resistance theory corresponds to the theory as discussed above, and to check this theory we have a ship model towing

tank. Was there a close relationship between these two in the past, though? Unfortunately, we must admit that an effective and complementary cooperation such as found in other fields of science, for example, the close cooperation found between the wing theory or boundary layer theory and the wind tunnel in aeronautics is not found. How has this come about?

In the next section a few plausible causes will be discussed.

B. Usefulness and Limit of Resistance Tests

First of all, the resistance test is not a well suited method to find an optimum hull form, although it may be an effective means of confirming the resistance of a given ship. It is essentially a method of confirmation but not a method to seek the best hull form. We should have another experimental method to attain the true aims of the towing tank. A paper read at the Society of Naval Architects and Marine Engineers meeting last fall (1962) by one of the authors, Takao Inui, also stresses this point.

The existing method of testing hull forms utilizes as its principal technique the "measurement of force," which provides only integral values, that is, the total resultant of wave-resistance as obtained experimentally and indirectly. To a more searching and fundamental question concerning the processes through which wave-resistance is generated, the Froude's method fails to give an answer. The wave-making resistance can be understood in several ways, e.g. Michell's pressure integration,[2] Havelock's wave-energy integration,[3,4] or any one of the numerous methods for obtaining wave-resistance theoretically. As far as theory alone is concerned the best one among the above may not be easily determined a priori, but must be settled case by case. In connection with the towing tank work, which is almost the only means for such experimental checks, "observation and analysis of wave-pattern of ship model" or simple "wave-analysis" based on the Havelock's theory must be the most effective method, because the waves are the only phenomena which can be observed in experiments. There can not be any dispute about the superiority of this method.

In terms of the Havelock's theory described above, the process of analysis:

$\boxed{\text{HULL FORM}} \rightarrow \boxed{\text{SHIP WAVE}} \rightarrow \boxed{\text{WAVE-RESISTANCE}}$

constitutes a kind of a syllogism, and any fault or incompleteness in the theory an and must be corrected in the intermediate state of "ship wave", a physical phenomenon, by comparing the theory with experimental results, (Figure 1). This intermediate step could be skipped only when the theory would have progressed far enough that the boundary conditions on the hull and the free surface could be satisfied to a far greater degree of accuracy than what can be done today, and that the effect of non-linearity and viscosity with its entire complexity and subtlety could be successfully taken into consideration. It could aos be skipped if the relation between the hull form and the wave-resistance were so simple that the theory would need no correction in the intermediate step. Unfortunately, however, the wave-making phenomenon is not only complex but often a small change in the hull form brings a large change in wave-resistance -- ironically true at the relatively low Froude numbers of commercial ships.

From the above discussions, it must be clear that ship model testing should not terminate with a conventional measurement of forces such as found in resistance and propulsion tests, but should also include the new "wave-analysis" method of hull form tests. One point must be clarified at this point. The "wave-analysis" method stated above, concerns the whole wave-pattern. Wave measurement at the ship's side is of course a part of "wave-analysis," but it can never constitute the whole program. Moreover, only when we seriously consider the problem of obtaining a complete wave-pattern in a tank, do we come to realize the inadequacy and the biased features of the existing towing tanks and the shortcomings of the present hull form testing method. A ship model towing tank is physically very large compared to the experimental apparatus used in other fields of engineering, such as wind tunnels and electronics facilities, and it can not easily be changed.

This may, in part, explain why there has not been an effective cooperation between theory and experiment. For, on the theoretical side, the valuable contributions made by Havelock have been with us for a long time.

C. An Approach to the Ship Model Testing Method Based On the "Wave-Analysis" Adopted at the University of Tokyo Experimental Tank -- A General Description

The purpose of "wave-analysis" is to check and correct the incomplete aspects of wave-resistance theory based on the experimental evidence at the intermediate stage, "ship wave." "Wave-analysis" work,

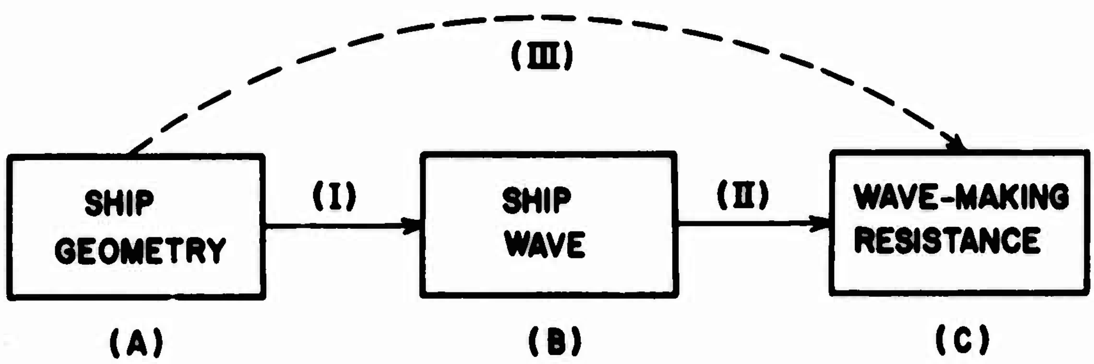


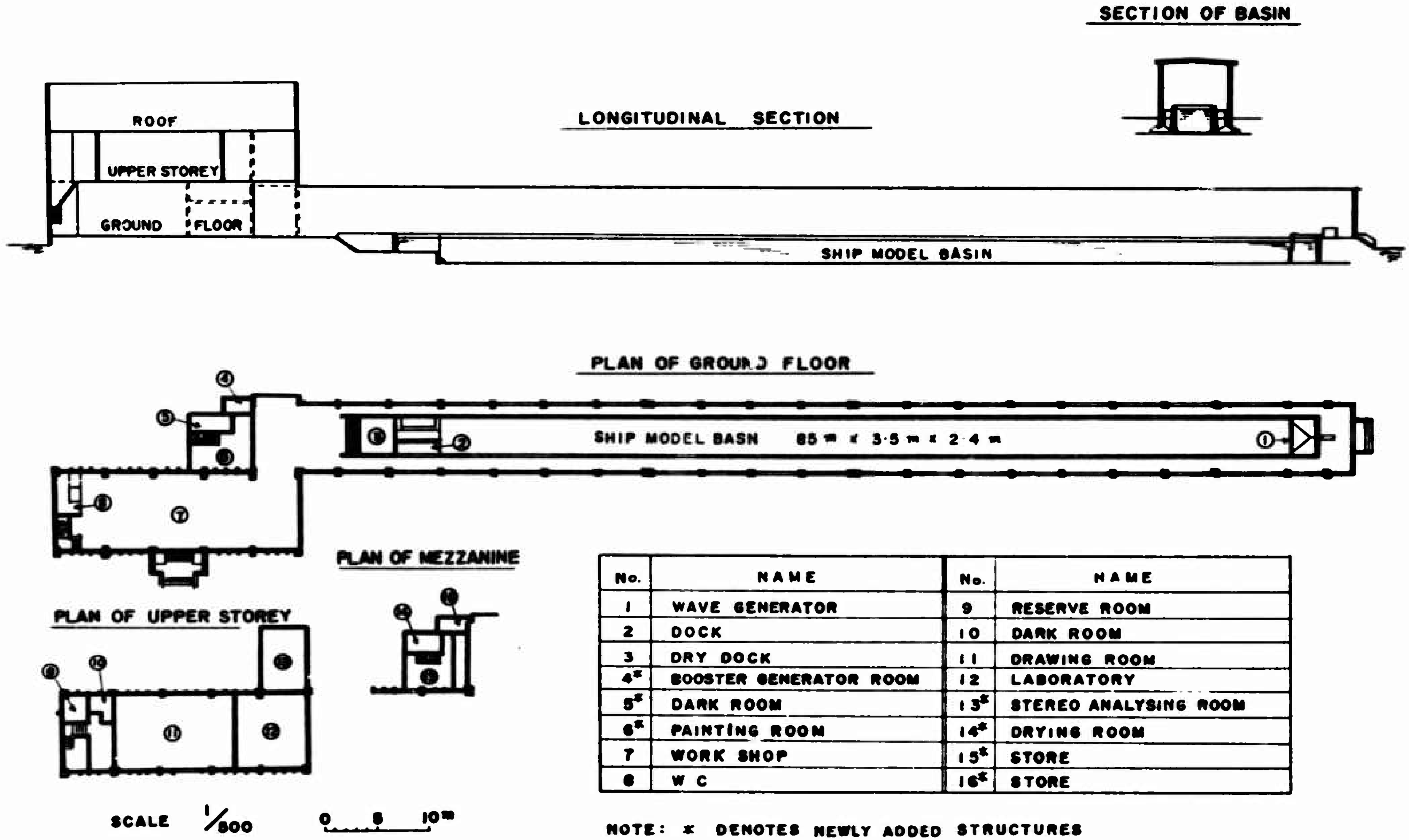
Figure 1

therefore, is divided into two aspects, one concerning theoretical calculation (especially wave profile calculations in every direction) and another concerning the techniques of wave observation. In what follows in this paper, experiences in the latter obtained at the University of Tokyo Experimental Tank is discussed.

First, Figure 2 is presented to give some idea of the present facility of The Experimental Tank (opened in 1937). On the same figure are marked the structures added since 1962 and finished in April 1963. From the beginning, we emphasized the importance of obtaining over-all wave pattern rather than the details and decided to develop the photographic method as the best suited for this purpose. We have had no reason in the last ten years to change our opinions. In the early stages, our only concern was to take as sharp and clear a photograph of the wave pattern of a model as possible. This stage was thought to be the most important and, has turned out to be, by far the most laborious problem that we have had to face. Compared to this, the second stage, which involved advancing from a single picture to a pair of stereophotos, in other words, proceeding from a qualitative observation to the quantitative measurement of waves, was fairly easily accomplished. For the study of hull forms, we are convinced that a panoramic view of the over-all wave pattern should go ahead of any quantitative stereo-analysis.

Besides the stereo-photo method, numerous other methods such as searching-probe have been devised to measure surface elevation. But, these other methods, except stereo-photos do not readily provide a qualitativé view of over-all wave patterns. For the over-all wave-pattern, inevitably photogrammetric vertical pictures become necessary. According to the experts of photogrammetry, if ideal conditions for taking pictures were provided, it would be possible to obtain an accuracy to within $\pm$ 0.1 mm in wave height measurement. In order to meet these conditions for quality photography, again considerable difficulty was encountered. As will be explained later in this section dealing with water surface lighting, putting up dark shades on all windows on both sides of the towing tank extending about 100 meters in length was by itself a costly task at the time.

Initially partial wave-patterns involving only a small area were photographed. As progress was made, the area was gradually extended to cover the complete wave pattern of a running model. Through these experiences, we gained a strong impression. Most existing towing tanks are very poorly built to take photographs of wave patterns. Everything is for "force measurement" with the belief that that is the



No.	NAME	No.	NAME
1	WAVE GENERATOR	9	RESERVE ROOM
2	DOCK	10	DARK ROOM
3	DRY DOCK	11	DRAWING ROOM
4*	BOOSTER GENERATOR ROOM	12	LABORATORY
5*	DARK ROOM	13*	STEREO ANALYSING ROOM
6*	PAINTING ROOM	14*	DRYING ROOM
7	WORK SHOP	15*	STORE
8	W C	16*	STORE

NOTE: * DENOTES NEWLY ADDED STRUCTURES

Fig. 2: General Arrangement of Tokyo University Tank

whole thing to be done for any model testing, and as a consequence, difficulties in taking vertical photographs have resulted. Among other problems that had to be considered were the location and method of towing of the model and the location of the camera. These problems immediately become entangled with the structural configuration of the towing tank, in particular, the ceiling. If a careful consideration is not paid to all these problems in designing a twoing tank which is to be used for comprehensive research work on wave-making resistance, the structure becomes too difficult to alter and will ever remain as an unsolvable problem.

In the following, the experience obtained at The University of Tokyo in the past 10 years is summarized in a chronological order. We hope that this will be helpful to those who intend to attempt "wave analysis" in the existing towing tanks and to those who are engaged in construction of new tanks. For information concerning progress up to the stereophotos first obtained at The University of Tokyo late in 1960, refer to a paper by Takahei.(5) The present facility is essentially along the same lines as the one described in that paper.

D. An Approach to the Ship Model Testing Method Based on the "Wave-Analysis" at the University of Tokyo Experimental Tank -- A Detailed Description

D-1. The Method of Towing Models and the Design Problem of the Towing Carriage

Since routine model testing has relied solely on the measurement of force, the construction of the present carriage is set up for that type of measurement only. The resistance dynamometer is situated in the center and the model is attached underneath. The distance between the towing carriage floor and the water level is very small. Therefore, the model and the most important part of the wave pattern is hidden under the carriage. It makes it very difficult not only to take the vertical photographs, but also to observe the wave-pattern. This is perhaps the main reason why so far the study model wave patterns has referred only to the wave profile at the model side. In order to grasp the whole wave-pattern all at once, the carriage must be designed so that there must be an uninterrupted space around and especially aft of the model, and a camera must be placed directly above this open space. For this purpose, a second auxiliary carriage, or trailer, is provided besides the main carriage. The trailer is fastened to the main carriage by a suitable means to retain an opening for wave observation, open space being 6 meter in length and slightly longer than twice the length of the model

(Figures 3 and 4). It is to be noted in passing that the principle dimensions of the University of Tokyo Experimental Tank are 85 (length) x 3.5 (width) x 2.4 (maximum depth) in meters. The standard model length is 2.5 meters except in the case of tankers, in which case a 2.8 - 3.0 meter model is used. As shown in Figure 3, the ship model is placed as far ahead as possible in this open area, that is right behind the main carriage. To place stereo cameras (three sets) directly above the centerline and also to connect the two carriages, a horizontal girder is placed 3.4 meters (the maximum allowable height) above the water surface as shown in Figure 3. The setup shown in Figure 3 was completed in March 1962. It was designated by T. Tagori and constructed by Yokohama Shipyard and Engine Works, Mitsubishi Nippon Heavy Industries, Ltd. Tagori's design followed an earlier setup designed by T. Takahei which had been used for approximately three years. Takahei's design had two cameras mounted on a centerline cantilever beam. The experience obtained from this setup indicated the necessity for several changes, which are now in Tagori's design. Special attention was given to the problems of vibrations, deflections and lightness of structure in the design of the new girder. Previously, a simple carriage made of wood had been used for the auxiliary carriage, but a completely new carriage has been made, which is provided with a wider wheel base and a more rigid main girder constructed of steel. The trailer and the centerline girder for the cameras can be detached from the main carriage if necessary. This can be done in 2 or 3 hours.

D-2. Performance of Stereo-Cameras and the Ceiling Height in the Towing Tank

The accuracy of measurement in wave height increases with increasing distance between the two cameras. (This distance is called base length. On the other hand, the overlap is usually about 50%, which is the portion of the object plane which appears on both photographs. Therefore, it becomes advantageous to have the camera as high a place as possible in proportion to the model's length. In existing towing tanks, this kind of consideration has been left aside. At The University of Tokyo the situation is slightly better than the bigger tanks, because of the smaller size of our modes, but it has barely 3.4 meters clearance above the water level, which is almost the minimum required height for the camera. This height is hardly sufficient for the standard model length (2.5 - 3.0 meters) and this fact has imposed various difficult requirements on the resolution qualities of lens and the lighting technique. With these considerations in mind, we recommend a ceiling height of at least 1.5 times the length of the maximum model to be tested.

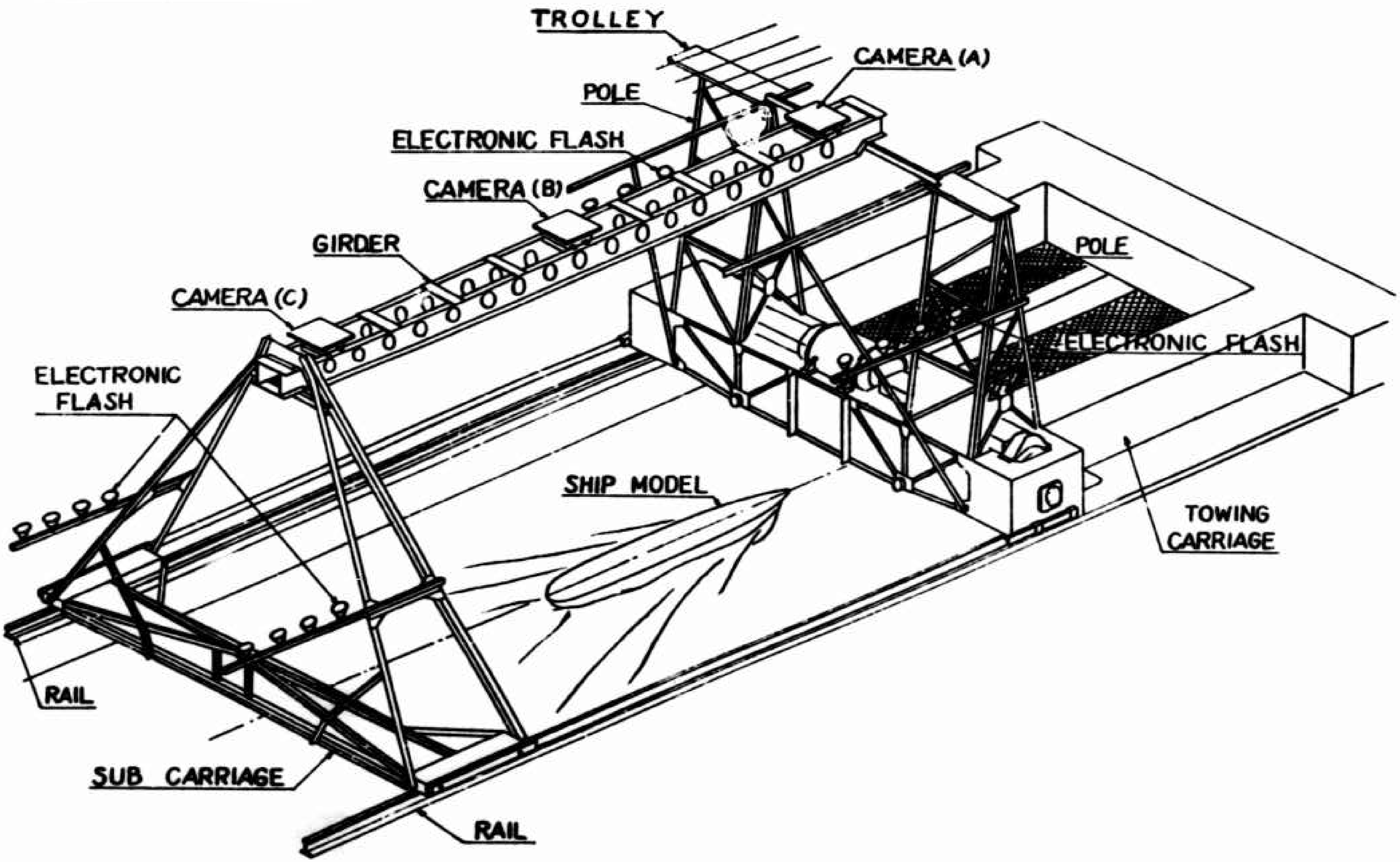


Fig. 3: Sketch of Arrangement of Photogrammetry of Ship Model's Wave Pattern

The problem of low ceiling height can be solved in two ways. One is to build a small model particularly for the study of wave analysis. This, however, must be avoided if possible. In the case of The University of Tokyo, the problem of ceiling height was fortunately not critical enough to resort to this last measure. The problem was solved temporarily by using an extremely wide-angle lens. In general, it must be remembered that the use of extremely wide-angle lens introduces distortion and as a consequence such a lens can not be used with too large an aperture. Suppose the range of the coverage should be at least twice the model length, for a camera to be located at a distance of 3.4 meters, it was decided that the widest-angle lens manufactured and obtainable in Japan, a 110°lens, was needed. As for the brightness of the lens, f = 12 was guaranteed at the beginning but actually only f = 32 was obtainable with the required accuracy in the resolution of about 30 lines per millimeter. The focal distance of the lens is 132 mm.

The most severe restriction was imposed on the camera located in the center Camera (B) in Figures 3 and 4. Distortion correction can be made for each lens by using a transparent pressure plate, which may also be used to flatten out the film, and which is calibrated for the distortion characteristics of each lens. Film size is 230 x 230 mm, a standard aerial survey type. With a 110° lens the camera angle is about 84° to parallel sides of the film. Taking the picture from a 3.4 meter height, the coverage span in the direction of parallel sides is 6.140 meters. These relationships are shown in Figure 4. As is shown in the figure, wave analysis can be performed over the area from the F.P. to the A.P., using Cameras (A) and (B), and over an area of a ship's length behind the model, using Cameras (B) and (C). The accuracy in wave height measurement should ideally be about ± .1 mm as explained in Section C, but the present resolution has not reached that figure. The main reasons for this is the difficulty of confirming pass points in analyzing the stereo-pair pictures. The pass points (usually a minimum of four) define the datum plane at a known height above the water surface, from which the plane contours of the undulating water surface can be determined. The pass points are presently not fixed to a stationary system but to the carriage, and hence is vulnerable to the carriage vibrations. The second reason is because of difficulty in pinpointing a location in two different photographs. We have not yet been able to devise a satisfactory method by which we could clearly identify a point on a pair of photographs.

We feel that stationary points would be preferable to moving points, and we plan to change to this stationary system after the installation of an automatic speed control for the carriage which will be discussed later. In regard to the latter difficulty, we are

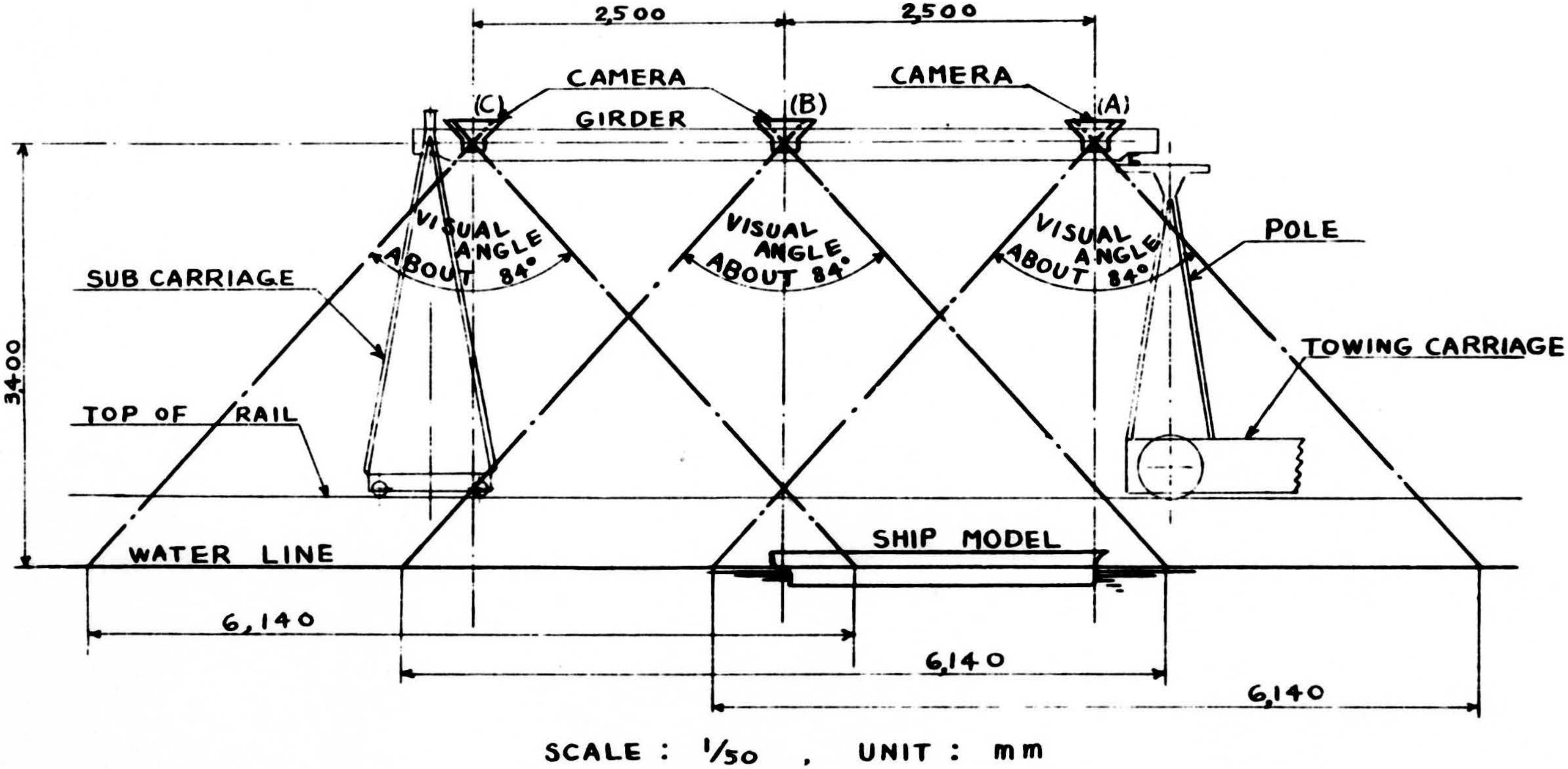


Fig. 4: Arrangement of Photogrammetry of Ship Model's Wave Pattern

currently trying out several alternatives. One is to deliberately overspray aluminum powder, and the other is to spray fine wool tuffs or sawdust particles which would float on the surface and would become better photographing targets.

The specification for the stereo cameras described in the above was prepared by T. Takahei and the cameras were manufactured by Kokusai Aerial Survey Company, Ltd., Japan. The cost was borne by a part of 1961 research fund provided by the Japan Shipbuilding Research Association, 45th Research Committee (A Study on Hull Forms of High Speed Liners).

D-3. Illumination of Water Surface

Besides the difficulties already discussed in Section D-1, special illumination conditions are required for taking vertical photographs in the towing tank. After numerous trials of different methods, we are now using Takahei's method of spraying fine aluminum powder (mesh 150-170) and indirect illumination by using strobe lights (flash time 1/2000 sec.). Arrangements are as shown in Figure 3. The number of strobe units is 4 x 4 = 16. In illumination a somewhat troublesome problem has arisen. This is because of the extremely wide-angle lens already described in Section D-2. For an extremely wide-angle lengs, the flux of light beams passing through the center of the lens and that through the peripheral of the lens is quite different in quantity. If only one camera is used, this problem can be solved by suitably adjusting lights. However, when two and three cameras are used simultaneously, a part of the wave-pattern that may be a central image for Camera (B) becomes peripheral images for Cameras (A) and (C). As a result, it becomes almost impossible to adjust the lights for uniform images. Under these conditions, ideal photographs are difficult to obtain. The best solution to this problem is again to have a high ceiling. If this is not possible, the next best method is to artificially apply a darkening coating in the center of the lens so as to bring the brightness of the center par with that of periphery Dark coating in the center of the lens, however, can only be applied to fast lenses. At present there seems to be no margin to spare in the brightness of the lens. Whether to increase the number of strobe units alone or to use the coating method with increased strobe units is currently being studied.

The water surface must be indirectly illuminated by fully dispersed beams to avoid glare and hence all strobe units must be facing the ceiling. Therefore, the reflection qualities of the ceiling become important. An absence of windows in a towing tank is preferable in order to avoid outside light sources.

D-4. Towing Carriage Speed Control System

Sections D-1 through D-3 describe the approximate status of The University of Tokyo Experimental Tank up until March 1962. The more fundamental problems, however, had been left untouched. We were especially aware of the need for the carriage speed control device described in this section and the stereoplotter described in Section D-5, but because of the fairly large expense involved, these problems could not be solved within the limit of the annual budget. At this point, fortunately a grant of about 31,900 American dollars was donated from the Tōyō Rayon Foundation for the promotion of Science and Technology in March 1962. In connection to this, the Ministry of Education provided subsidies of about $6,600 for the new structure to enclose the new facilities and equipments and about $2,000 for the research fund. The latter two amounted to $8,600. Thus a total of $40,500 was obtainable from the government and private sources. This fund is proportionately only about 10% or less of the initial construction cost of The University of Tokyo Experimental Tank (currently adjusted cost is approximately $420,000 to $550,000), but for the purpose of improving the facility, that is, to have a model testing based on the new "wave analysis," the fund is ample enough.

Since April 1962, in The University of Tokyo Experimental Tank, staffs have been busily engaged on the works dealing with the new construction, carriage speed control devices and stereoplotter in addition to the long range research projects concerning "Basic Study on Wave Resistance of Ships and Its Practical Applications to Hull Form Designs" which had been going on for these ten years. Among these, construction of the new structure was given a top priority and was finished by April 1963 (see Figure 2).

The fund from the Tōyö Rayon Foundation was mainly used for the installation of the carriage speed control device described below and stereoplotter described in Section D-5. T. Tagori was responsible for the design of carriage speed control device and was assisted by H. Kajitani. The principal dimensions and capacity of the carriage speed control are as follows:

(a) Carriage speed selection

accuracy	± 1 mm/sec or less
steps	1 mm/sec

(b) Speed regularity ± 1 mm/sec root mean square

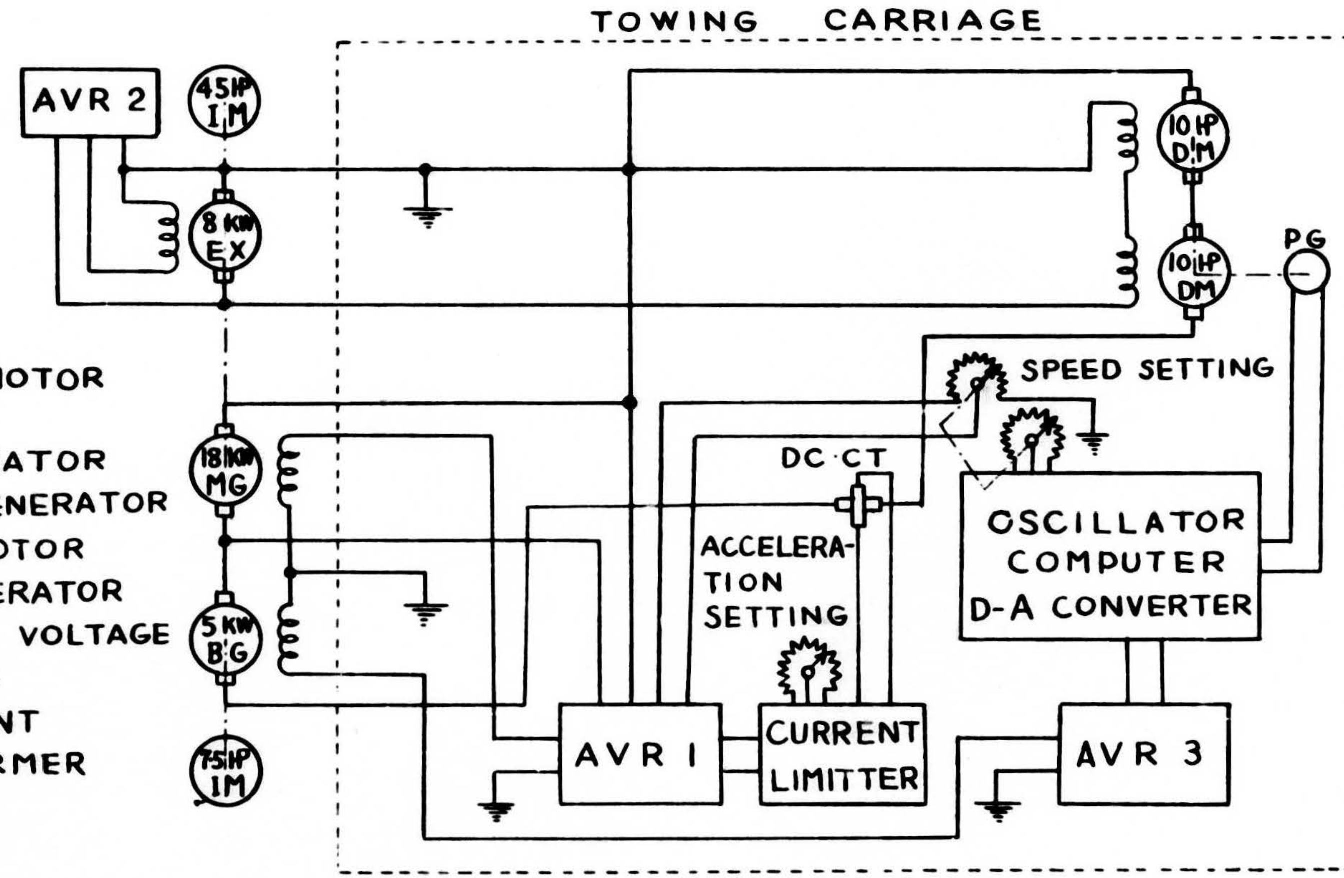
(c) Range of speed control 0.5 - 2.0 m/sec.
Furthermore, approximately similar degree of selection should be available for the whole speed range of 0.3 - 4.0 m/sec.

(d) Time required to attain steady speed.
3 seconds or less to steady state after the end of acceleration.

(e) Acceleration and deceleration.
Changeable within the range of 0.01 g - 0.1 g. Should not exceed the maximum of 0.1 g. Degree of selection: less than $\pm$ 10%.

(f) General description of this apparatus:
Block diagram is given in Figure 5. With this apparatus the carriage is accelerated at a constant acceleration, run for a desired time at a steady selected speed and reverse control with a constant deceleration to a stop.

Special emphasis is placed on the first item (a), a fine selection of speeds. This is based on the following reasons. As is described at the beginning of Section C, a complete "wave-analysis" process includes initial calculation of wave-profile in every direction, experimentation including a detail account of where the theory is correct and where it is not. In particular the last step, comparison between the theory and the experiment, is an important one. Calculation of wave profile, however, requires a great deal of labor and many previously arranged numerical tables. These tables are made up with suitably selected variables for certain specified Froude numbers. Therefore, if the carriage speed does not correspond to the Froude number available in the table, the comparison necessarily becomes much more involved. Ideally the model would be run exactly at a speed for which the table is prepared. It is this fact which necessitates a different speed control device in our case from those required only for the conventional resistance and propulsion tests. The predetermined speed selection is also needed for the comparative analysis of the so-called differential wave profiles. For example, to a given hull an appendage such as a bulb is added without modifying the main hull. Under the assumption of linear superposition, one can experimentally study the waves generated by the appendage (bulb in this case) by comparing the difference between the wave profiles of main hull plus bulb and main hull only. The set of runs must, therefore, necessarily be performed at the same speed. And furthermore, since



IM : INDUCTION MOTOR
EX : EXCITER
MG : MAIN GENERATOR
BG : BOOSTER GENERATOR
DM : DRIVING MOTOR
PG : PULSE GENERATOR
AVR : AUTOMATIC VOLTAGE REGULATOR
DC·CT : DC CURRENT TRANSFORMER

Fig. 5: Block Diagram for Automatic Speed Controller for Towing Carriage

this "measured" differential wave profile is to be compared to the calculated value for the bulb, each of the runs should be done at an exactly preselected speed.

At The University of Tokyo Experimental Tank the carriage speed control device had been an old constant voltage type which has given an extreme amount of difficulty in this respect so that the numbers of carriage runs had to be close to twice those of successful runs right at the intended speeds. We are, therefore, now very eagerly awaiting the completion of this control device which is expected to be finished in September 1963. Construction is being carried out by the Nippon Electric Company, Electronics Joint Division. From the beginning Mr. Chikara Arai, Research Staff at the Ship Research Institute, has kindly offered many technical guidances and suggestions to this problem.

D-5. Stereoplotter

The stereo-pair photographic films have so far been sent out to Kokusai Aerial Surveys Co. Ltd. to be analyzed. We felt the need for doing photoanalysis by ourselves, in particular for a speedy reduction of data. Among several types of plotters available, the one that can be used with only a short practice and which has a fairly high degree of accuracy, a Kelsh plotter, a double projection type belonging to the second class of plotters was purchased from the above company. It was completed in June 1963, and a photograph is shown in Figure 6. Its special feature is use of anagliph stereoscopy. Anagliph stereoscopy takes stereophotographs of a three dimensional body and projects on the screen in two colors. From the image the body is retraced. Specifications for the stereoplotter are as follows:

(a) Wide angle projector: distance to the screen
  Principal distance 150 - 159 mm
  Film size 230 x 230 mm

(b) Minimum base length: 260 mm

(c) Tilt $\pm 5^\circ$

(d) Over-all tilt $\pm 5^\circ$

(e) Height of projection 680 mm ($= \frac{1}{5} \times 3400$ mm)

(f) Scale of position 1/6

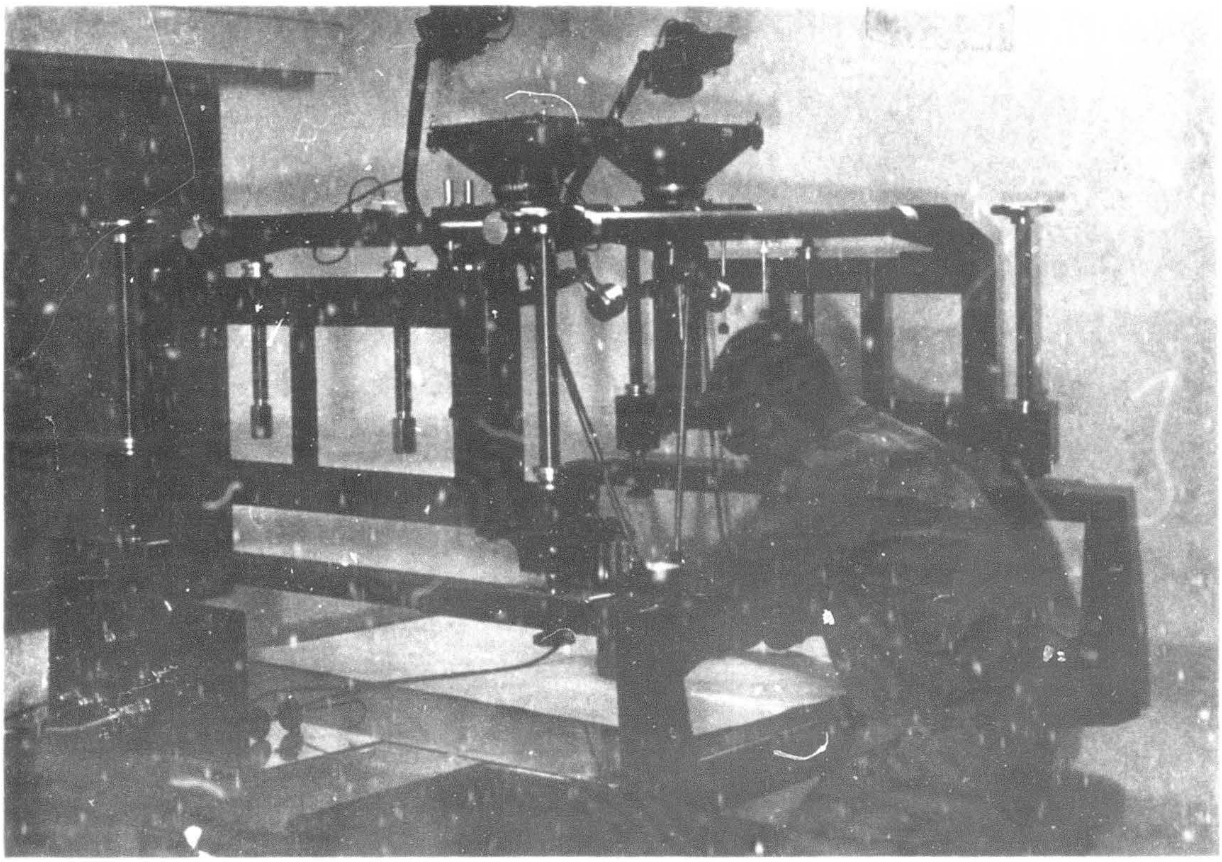


Fig. 6: Stereoplotter

(g)	Scale of height	1/5
(h)	Tracing table height	100 mm $\pm$ 100 mm
(i)	Easel	1200 x 1400 mm
(j)	Projection lamp	12 volts 50 watts (from AC 100 V.)
(k)	Total weight	390 kg.

D-6. Miscellaneous

For theoretical calculations of wave-resistance problems, it is already a common practice to use a computing machine as similar to the IBM 7090. However, in a towing tank, a small electronic computer is needed for preliminary calculations and for some intermediate calculations before the data is put into large machines. Presently we are considering the installation of such a machine.

A large room for drying films of unusually large size (230 x 230 mm) and a fairly large dark room was needed to handle the stereo-photographs. Fortunately, this problem was solved with the new addition of previously mentioned structures.

ACKNOWLEDGMENTS

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APPENDIX

After a trial of the new facilities mentioned in the contents, several modifications of the photographic apparatus have been undertaken; these are in the camera lens and stereo-projector.

The new characteristics of the apparatus are shown in Table A compared with those of the previous ones.

1. Camera and Lens

The f-number of the new lens is f = 9 compared with f = 32 for the old one. The brightness is improved as much as 11 times. The film holder is changed to the type in which the film is pressed to the camera's focal plane by use of a vacuum. With the lens modification more brightness around the circumference, in other words more uniform brightness of a picture is expected. The film holder modification improves the accuracy of the principal distance, which means the higher accuracy of the analysis involved.

2. Projector

The principal distance of the projector is increased to 211.0 mm from 158.4 mm. The height in this case is exaggerated so as to render a higher accuracy. The ratio of the accuracy of position to that of height is now improved as much as 60% compared with the previous projector which had about 20% improvement over the case using the same principal distance as the camera lens.

TABLE A

CAMERA

	Previous Camera	New Camera	Used as Aerial Camera
Principal distance, mm	$f_c = 132$	"	$f_c = 153$
Flight height, mm	$h = 3400$	"	H
Base length, mm	$b = 2500$	"	B
Base height ratio	$b/h = 1/1.36$	"	$B/H = 2/3 = 1/1.5$
Scale of photograph	$M_c = f_c/h = 1/25.8$	"	
f - number	$f = 32$	$f = 9$	
Film holder	Regular	Pressed to focal plane	

PROJECTOR

		Previous Projector	New Projector
Principal distance	f_p, mm	158.4	211
Height of projection	h , mm	680	907
Magnification of projection	$M_p = h_p/f_p$	4.29	4.29
Scale of position	$M_{xy} = M_p M_c$	1/6	1/6
Scale of height	$M_z = f_p/f_c - M_{xy}$	1/5	1/3.75

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AN EVALUATION OF THE METHOD OF DIRECT DETERMINATION OF WAVEMAKING RESISTANCE FROM SURFACE-PROFILE MEASUREMENTS

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AN EVALUATION OF THE METHOD OF DIRECT DETERMINATION OF WAVEMAKING RESISTANCE FROM SURFACE-PROFILE MEASUREMENTS

Introduction

Several methods have recently been proposed for determining the wavemaking resistance of a ship from measurements of the profile of the free surface.[1,2,3] A comparison between the results of such measurements and the wavemaking resistance obtained by the usual procedure of subtracting from the total the viscous part of the resistance, given by an assumed correlation line or by wake surveys, suggests that the results obtained by the method of surface profile measurements may be seriously in error.

Two possible causes of error, one physical and the other mathematical, are considered in the present contribution. Since the theories of these proposed methods all assume that the fluid is inviscid and irrotational, it is important to discuss the error incurred when some of the surface-profile measurements are taken in the ship's wake. Secondly, because a certain approximate form of the velocity potential is assumed in some of the methods, it appeared desirable to compare the assumed form with the complete solution in a simple case. For this purpose the surface disturbance due to a source in a channel has been treated.

Effect of the Wake

In a recent publication[4] it has been shown that the total drag D of a ship form and its viscous drag D_V are given by the expressions

$$D = \int_S [p_o - p + \rho u(U-u)] \, dS + \frac{1}{2} \gamma \int_{-b}^{b} Z^2 \, dy \tag{1}$$

$$D_V = \int_\omega \{p_o - p + \rho u(U-u) + \frac{1}{2} \rho \, [(U-u_1)^2 - v_1^2 - w_1^2]\} dS \tag{2}$$

where p_o is the pressure and U the stream velocity at a great distance upstream from the hull, p is the pressure and u the longitudinal component of the velocity at a transverse section S at a moderate distance behind the model, ρ is the mass density of the fluid and γ its specific weight, b is the width of the channel, Z is the displacement of the free

surface from its undisturbed level, and u_1, v_1, w_1 are the velocity components within the wake region ω of the analytical continuation of the potential flow outside the wake. By defining the wavemaking resistance D_w by

$$D_w = D - D_v$$

Equations (1) and (2), together with the Bernoulli equation, were applied in[4] to derive the expression for D_w

$$D_w = \frac{1}{2} \rho \int_S [v_1^2 + w_1^2 - (U-u_1)^2] \, dS + \frac{1}{2} \gamma \int_{-b}^{b} Z^2 \, dy \qquad (3)$$

in which, outside the wake, u_1, v_1, w_1 coincide with the actual velocity components.

On the assumption that the fluid is inviscid and the flow irrotational, Wehausen[5] and Eggers[1] have obtained the same result[3] for D_w. This indicates that the methods for obtaining D_w from surface-profile measurements should be valid if the measurements within the wake were replaced by their inviscid, potential-flow counterparts in evaluating the surface integral in (3). Unfortunately, the velocity field u_1, v_1, w_1 and the corresponding surface displacement Z_1 within the wake are unknown, so that, practically, it would be necessary to estimate Z_1 by interpolation from the measured values of Z outside the wake. Studies of mathematically simulated wake flows, generated by source distributions of non-zero total strength, may indicate the type of interpolation that should be made. If the contribution from the values within the wake is a small part of the total, a rough interpolation for the values of Z_1 may be found to suffice. It should be emphasized that these remarks apply only to the analysis of the surface integral in (3); in the last term in (3) the actual values of $Z(x,y)$, including those measured in the wake are to be used.

Surface Disturbance due to a Source

As was pointed out in [4], associated with the potential flow velocity components u_1, v_1, w_1 there is a source distribtuion within the wake. Thus it is conceivable that, even at several model lengths downstream from a ship form, it may be necessary to take into account the field of neighboring sources in estimating the values of Z_1. Since the form of the velocity potential assumed in [1] neglects a part which is

appreciable only in the region close to the wave-generating body, the possibility exists that the error due to the neglect of the "near-field" potential of these wake sources may be large.

In order to investigate the relative magnitudes of the surface disturbances due to the near- and far-field potentials, let us consider the case of a source of unit strength situated at a depth c below the free surface halfway between the vertical walls of an infinitely deep channel. The coordinates of the source will be taken to be $(0, 0, -c)$ relative to a righthanded Cartesian coordinate system with the x-axis in the free surface along the centerline of the channel and the z-axis with its positive sense vertically upwards. The channel walls will be taken to be the planes $y = \pm b/2$. It will be supposed that the source is moving with velocity $\bar{U}$ in the positive sense of the x-axis.

The velocity potential of a unit source in a channel of unlimited extent, for a source located at $(0, y_s, -c)$, is expressible as the real part of the form(6)

$$\phi_s = \frac{1}{r'} - \frac{1}{r} + \lim_{\mu \to 0} \frac{k_o}{\pi} \int_{-\pi}^{\pi} \sec^2\theta \, d\theta \int_0^\infty \frac{e^{k(z-c)+ik[x \cos \theta + (y-y_s) \sin \theta]}}{k-k_o\sec^2\theta+i\mu \sec \theta} dk \tag{4}$$

in which

$$r^2 = x^2 + (y-y_s)^2 + (z+c)^2, \qquad r'^2 = x^2 + (y-y_s)^2 + (z-c)^2$$

$$k_o = \frac{g}{U^2}$$

and g is the acceleration of gravity. The equation of the disturbed free surface, $z = Z(x,y)$, can be obtained from the velocity potential ϕ by means of the relation

$$Z(x,y) = \frac{U}{g} \left(\frac{\partial\phi}{\partial x}\right)_{z=0} \tag{5}$$

Hence the disturbance Z_s associated with ϕ_s is given by

$$Z_s = \frac{1}{\pi U} \int_{-\pi}^{\pi} \int_0^{\infty} \frac{e^{-kc+ik[x \cos\theta + (y-y_s)\sin\theta]}}{k-k_o \sec^2\theta + i\mu \sec\theta} k \sec\theta \, d\theta \, dk \tag{6}$$

where it is understood that μ is to be set equal to zero in the final result.

The boundary condition of zero normal velocity on the walls at $y = \pm b/2$ is satisfied by superimposing on the field due to the source at $(0, 0, -c)$ that due to the array of image sources at $(0, nb, -c)$, $n=\pm 1$, ± 2, ... The resulting equation for the surface disturbance $Z(x,y)$ can then be written in the form

$$Z = \lim_{N\to\infty} \frac{1}{\pi U} \sum_{n=-N}^{N} \int_{-\pi}^{\pi} \int_0^{\infty} \frac{e^{-kc+ik[x \cos\theta + (y-nb)\sin\theta]}}{k-k_o \sec^2\theta + i\mu \sec\theta} k \sec\theta \, d\theta \, dk \tag{7}$$

Applying the easily verified formula

$$\sum_{n=-N}^{N} e^{-in\alpha} = \frac{\sin(N+1/2)\alpha}{\sin \alpha/2}$$

we obtain

$$Z = \lim_{N\to\infty} \frac{1}{\pi U} \int_{-\pi}^{\pi} \int_0^{\infty} \frac{e^{-kc+ik(x\cos\theta + y\sin\theta)} \sin[(N+1/2)kb\sin\theta]}{(k-k_o \sec\theta + i\mu\sec\theta)\sin(1/2\, kb \sin\theta)} k \sec\theta \, d\theta dk \tag{8}$$

The double integral in (8) may be considered to extend over an entire plane in which (k,θ) represent polar coordinates and $k \, d\theta \, dk$ an element of area. Hence, transforming to rectangular coordinates,

$$\xi = k\cos\theta , \qquad \eta = k \sin\theta$$

one obtains from (8)

$$Z = \lim_{N\to\infty} \frac{1}{\pi U} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-c\sqrt{\xi^2+\eta^2} + i(x\xi+y\eta)} \sin[(N+1/2)b\eta]}{(\xi - \frac{k_o}{\xi}\sqrt{\xi^2+\eta^2} + i\mu) \sin \frac{1}{2} b\eta} d\xi \, d\eta \tag{9}$$

Let us now employ the Dirichlet formula

$$\lim_{N\to\infty} \int_{-a}^{a} f(p)\,\frac{\sin(Np)}{p}\,dp = \pi f(0) \tag{10}$$

from which one can obtain as a corollary the result

$$\lim_{N\to\infty} \int_{-\infty}^{\infty} f(p)\,\frac{\sin[(2N+1)p]}{\sin p}\,dp = \pi \sum_{m=-\infty}^{\infty} f(m\pi) \tag{11}$$

provided $f(p)$ is continuous and has only a finite number of maxima and minima, and the last series converges. This result, applied in (9), yields

$$Z = \frac{2i}{Ub}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-c\sqrt{\xi+\frac{4\pi^2m^2}{b^2}}+i\left(x\xi+\frac{2\pi my}{b}\right)}}{\xi^2-k_o\sqrt{\xi^2+\frac{4\pi^2m^2}{b^2}}+i\mu\xi}\,\xi\,d\xi \tag{12}$$

In order to evaluate the integral in (12) it will be useful to replace ξ in (12) by the complex variable $\zeta = \xi + i\xi'$. Multiplying the numerator and denominator of the integrand by

$$\zeta^2 + k_o\sqrt{\zeta^2+\frac{4\pi^2m^2}{b^2}} + i\,\mu\,\zeta$$

gives in the denominator the fourth-degree polynomial

$$\zeta^4 + 2i\mu\,\zeta^3 - (k_o^2+\mu^2)\zeta^2 - \frac{4\pi^2m^2}{b^2}\,k_o^2$$

If μ is neglected, this polynomial has the zeros

$$\zeta_{o1} = \sqrt{\frac{k_o(k_m+k_o)}{2}},\quad \zeta_{o2} = -\sqrt{\frac{k_o(k_m+k_o)}{2}};\quad \zeta_{o3} = i\sqrt{\frac{k_o(k_m-k_o)}{2}},$$
$$\zeta_{o4} = -i\sqrt{\frac{k_o(k_m-k_o)}{2}} \tag{13}$$

where $k_m = \sqrt{k_o^2+\frac{16\pi^2m^2}{b^2}}$. Of these, ζ_{01} and ζ_{02} are zeros of the

original denominator of the integrand in (12), ζ_{03} and ζ_{04} corresponding to the multiplicative factor. Second approximations, correct to terms of order μ, can now be obtained by substituting ζ_{01} or ζ_{02} for ζ in the second term of the polynomial. In this way one obtains the zeros

$$\zeta_1 = \zeta_{01} - \frac{i\mu}{2k_m}(k_m+k_o) \qquad \zeta_2 = \zeta_{02} - \frac{i\mu}{2k_m}(k_m+k_o) \tag{14}$$

When $m = 0$, these expressions become

$$\zeta_1 = k_o - i\mu, \qquad \zeta_2 = -k_o - i\mu \tag{15}$$

The integrands in (12) also have branch points at $\zeta = 0$ and $\zeta = \pm 2m\pi i/b$. In selecting the branch of $\sqrt{\zeta^2+4\pi^2m^2/b^2}$, one must bear in mind that this quantity is positive for both positive and negative real values of ζ, including the case $m = 0$.

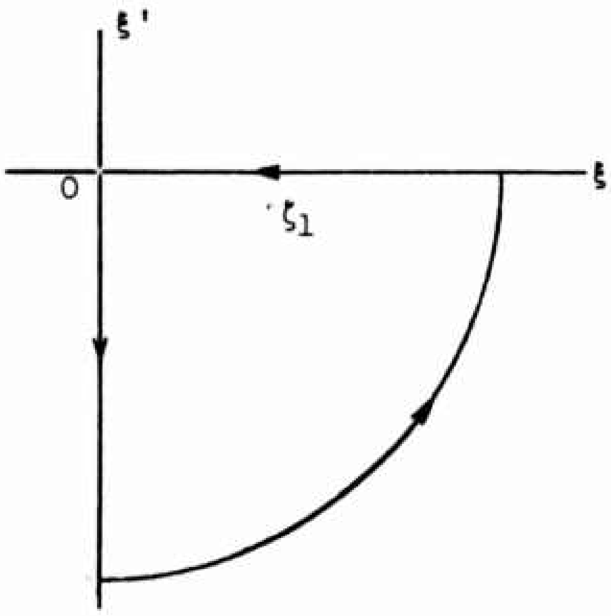

Figure 1a.

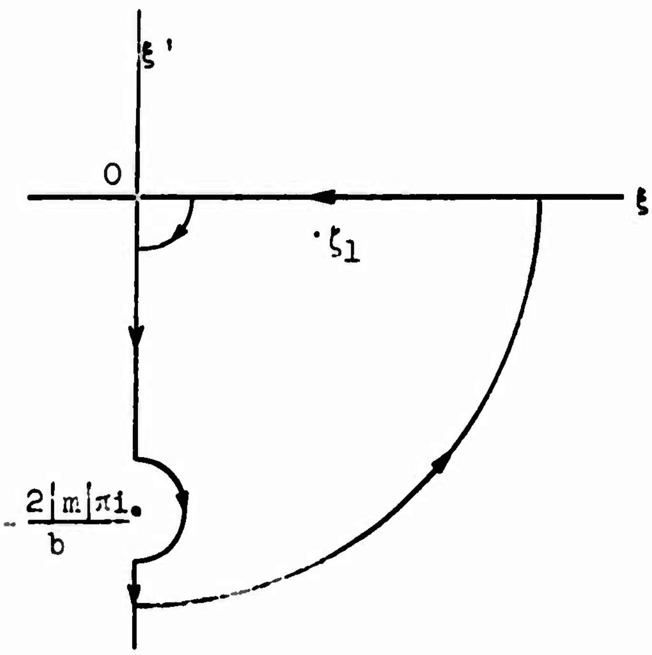

Figure 1b.

The term of the series in (12) corresponding to $m = 0$ may be written in the form

$$\int_{-\infty}^{0} \frac{e^{c\xi + ix\xi}}{\xi + k_o + i\mu} d\xi + \int_{0}^{\infty} \frac{e^{-c\xi + ix\xi}}{\xi - k_o + i\mu} d\xi \qquad (16)$$

First consider

$$\oint \frac{e^{-c\zeta + ix\zeta}}{\zeta - k_o + i\mu} d\zeta$$

evaluated over the contour shown in Figure 1a. For values $x \leqq 0$, the integral over the perimeter of the large circular quadrant vanishes as its radius approaches infinity, and hence we obtain

$$\int_{0}^{\infty} \frac{e^{-c\xi + ix\xi}}{\xi - k_o + i\mu} d\xi = i \int_{0}^{-\infty} \frac{e^{-x\xi' - ic\xi'}}{i\xi' - k_o + i\mu} d\xi' - 2\pi i \, e^{-c\zeta_1 + ix\zeta_1}$$

or, substituting $\xi' = -\xi$ in the second of the above integrals and letting $\mu \to 0$,

$$\lim_{\mu \to 0} \int_{0}^{\infty} \frac{e^{-c\xi + ix\xi}}{\xi - k_o + i\mu} d\xi = \int_{0}^{\infty} \frac{e^{\xi(x + ic)}}{\xi - ik_o} d\xi - 2\pi i \, e^{k_o(-c + ix)} \qquad (17)$$

It is redily verified that the first integral in (16) is the negative of the conjugate of the second. Consequently, the sum of the integrals in (16) is given by

$$2i \int_{0}^{\infty} \frac{e^{x\xi}[\xi \sin \xi c + k_o \cos \xi c]}{\xi^2 + k_o^2} d\xi - 4\pi i \, e^{-k_o c} \cos k_o x \qquad (18)$$

For the case $m > 0$ let us consider

$$\oint \frac{e^{-c\sqrt{\zeta^2 + \frac{4\pi^2 m^2}{b^2}} + i(x\zeta + \frac{2\pi m y}{b})}}{\zeta^2 - k_o\sqrt{\zeta^2 + \frac{4\pi^2 m^2}{b^2}} + i\mu\zeta} \zeta \, d\zeta$$

evaluated over the contour shown in Figure 1b. As in the previous case, for values $x \leqq 0$ the integral over the perimeter of the large circular quadrant vanishes as its radius approaches zero, and there remains

$$\lim_{\mu \to \infty} \int_0^\infty \frac{e^{-c\sqrt{\xi^2 + \frac{4\pi^2m^2}{b^2}} + i(x\xi + \frac{2\pi my}{b})}}{\xi^2 - k_o\sqrt{\xi^2 + \frac{4\pi^2m^2}{b^2}} + i\mu\xi}\,\xi d\xi = \int_0^{\frac{2\pi|m|}{b}} \frac{e^{-c\sqrt{\frac{4\pi^2m^2}{b^2} - \xi^2} + x\xi + \frac{2\pi my}{b}i}}{\xi^2 + k_o\sqrt{\frac{4\pi^2m^2}{b^2} - \xi^2}}\,\xi d\xi$$

$$+ \int_{\frac{2\pi|m|}{b}}^\infty \frac{e^{i(c\sqrt{\xi^2 - \frac{4\pi^2m^2}{b^2}} + \frac{2\pi my}{b}) + x\xi}}{\xi^2 - ik_o\sqrt{\xi^2 - \frac{4\pi^2m^2}{b^2}}}\,\xi d\xi - \pi i \frac{k_o + k_m}{k_m} e^{-\frac{e}{2}(k_o+k_m) + i(x\zeta_1 + \frac{2\pi my}{b})} \tag{19}$$

We also have

$$\int_{-\infty}^0 \frac{e^{-c\sqrt{\xi^2 + \frac{4\pi^2m^2}{b^2}} + i(x\xi + \frac{2\pi my}{b})}}{\xi^2 - k_o\sqrt{\xi^2 + \frac{4\pi^2m^2}{b^2}} + i\mu\xi}\,\xi d\xi = -\int_0^\infty \frac{e^{-c\sqrt{\xi^2 + \frac{4\pi^2m^2}{b^2}} - i(x\xi - \frac{2\pi my}{b})}}{\xi^2 - k_o\sqrt{\xi^2 + \frac{4\pi^2m^2}{b^2}} - i\mu\xi}\,\xi d\xi$$

which is seen to be the negative of the conjugative of the integral evaluated in (19), but with y replaced by $-y$. Hence we have

$$\lim_{\mu \to 0} \int_{-\infty}^\infty \frac{e^{-c\sqrt{\xi^2 + \frac{4\pi^2m^2}{b^2}} + i(x\xi + \frac{2\pi my}{b})}}{\xi^2 - k_o\sqrt{\xi^2 + \frac{4\pi^2m^2}{b^2}} + i\mu\xi}\,\xi d\xi$$

$$= 2i \int_{\frac{2\pi|m|}{b}}^\infty \frac{e^{x\xi + \frac{2\pi myi}{b}}\,[\xi \sin(c\,\alpha_m) + k\,\alpha_m \cos(c\alpha_m)]}{\xi^4 + k_o^2\,\alpha_m^2}\,\xi d\xi$$

$$- 2\pi i \frac{k_o + k_m}{k_m} e^{-\frac{c}{2}(k_o+k_m) + \frac{2\pi myi}{b}} \cos\left[x\sqrt{\frac{k_o(k_o+k_m)}{2}}\right] \tag{20}$$

where

$$\alpha_m = \sqrt{\xi^2 - \frac{4\pi^2m^2}{b^2}}$$

Since (20) reduces to (18) when $m = 0$, the real part of (12) yields the expression for the surface disturbance

$$Z = -\frac{4}{Ub}\sum_{m=-\infty}^{\infty}\cos\frac{2\pi my}{b}\left\{\int_{\frac{2\pi|m|}{b}}^{\infty}\frac{e^{x\xi}[\xi^2\sin(c\alpha_m)+k_o\alpha_m\cos(c\alpha_m)]}{\xi^4+k_o^2\alpha_m^2}\xi d\xi\right.$$

$$\left.-\pi\frac{k_o+k_m}{k_m}e^{-\frac{c}{2}(k_o+k_m)}\cos\left[x\sqrt{\frac{k_o(k_o+k_m)}{2}}\right]\right\},\quad x\leqq 0 \qquad (21)$$

This can be expressed in nondimensional form by setting $b = 1$, and $U = 1$, which yields

$$Z = -4\sum_{m=-\infty}^{\infty}\cos 2\pi my\left\{\int_{2\pi|m|}^{\infty}\frac{e^{x\xi}\,[\xi^2\sin(c\alpha_m)+k_o\alpha_m\cos(c\alpha_m)]}{\xi^4+k_o^2\alpha_m^2}\xi d\xi\right.$$

$$\left.-\pi\frac{k_o+k_m}{k_m}e^{-\frac{c}{2}(k_o+k_m)}\cos\left[x\sqrt{\frac{1}{2}k_o(k_o+k_m)}\right]\right\},\quad x\leqq 0 \qquad (22)$$

in which now

$$\alpha_m = \sqrt{\xi^2-4\pi^2m^2},\qquad k_m = \sqrt{k_o^2+16\pi^2m^2},\qquad k_o = \frac{1}{F^2}$$

where F is the Froude number $U/\sqrt{bg}$. The integrals in (22) can be expressed more conveniently in terms of $c\alpha_m$ as the variable of integration. Making this substitution, we obtain the alternative form

$$Z = -4\sum_{m=-\infty}^{\infty}\cos 2\pi my\left\{\int_0^{\infty}\frac{e^{\frac{x}{c}\sqrt{\alpha^2+4\pi^2m^2c^2}}\,[(\alpha^2+4\pi^2m^2c^2)\sin\alpha+k_oc\alpha\cos\alpha]}{\alpha^4+\alpha^2c^2(k_o^2+8\pi^2m^2)+16\pi^4m^4c^4}\alpha d\alpha\right.$$

$$\left.-\pi\frac{k_o+k_m}{k_m}e^{-\frac{c}{2}(k_o+k_m)}\cos\left[x\sqrt{\frac{1}{2}k_o(k_o+k_m)}\right]\right\},\quad x\leqq 0 \qquad (23)$$

At great distances downstream from the source, $(|x| \gg c)$, the integrals in (23) become very small and the surface disturbance assumes the form

$$Z \approx 4\pi \sum_{m=-\infty}^{\infty} \frac{k_o+k_m}{k_m} e^{-\frac{c}{2}(k_o+k_m)} \cos 2\pi my \cos [x\sqrt{\tfrac{1}{2} k_o(k_o+k_m)}] \tag{24}$$

in accordance with the assumption in Reference 1. How quickly the contributions of the near-field terms to the surface disturbance become negligible is indicated for each m by the ratio of the near-field term to the amplitude of the far-field term in (23),

$$\rho_m(x) = \frac{k_m e^{\frac{c}{2}(k_o+k_m)}}{\pi(k_o+k_m)} \int_0^{\infty} \frac{e^{\frac{x}{c}\sqrt{\alpha^2+4\pi^2m^2c^2}}\,[\alpha^2+4\pi^2m^2c^2)\sin\alpha+k_oc\alpha\cos\alpha]}{\alpha^4+\alpha^2c^2(k_o^2+8\pi^2m^2)+16\pi^4m^4c^4}\,\alpha d\alpha \tag{25}$$

<u>Evaluation of $\rho_m(x)$</u>

In order to evaluate $\rho_m(x)$ it is convenient to employ the quantities

$$F_{mn} = \int_{n\pi}^{(n+1)\pi} \frac{\alpha(\alpha^2+4\pi^2m^2c^2)e^{\frac{x}{c}\sqrt{\alpha^2+4\pi^2m^2c}}\sin\alpha}{\alpha^4+\alpha^2c^2(k^2+8\pi^2m^2)+16\pi^4m^4c^4}\,d\alpha$$

$$G_{mn} = \int_{(n-\frac{1}{2})\pi}^{(n+\frac{1}{2})\pi} \frac{\alpha^2 e^{\frac{x}{c}\sqrt{\alpha^2+4\pi^2m^2c^2}}\cos\alpha\,d\alpha}{\alpha^4+\alpha^2c^2(k_o^2+8\pi^2m^2)+16\pi^4m^4c^4}$$

We have then

$$\rho_m(x) = \frac{k_m e^{\frac{c}{2}(k_o+k_m)}}{\pi(k_o+k_m)} [\sum_{n=0}^{\infty} F_{mn} + k_oc\,(\tfrac{1}{2} G_{mo} \sum_{n=1}^{\infty} G_{mn})] \tag{26}$$

Since the terms of the two series in (26) alternate in sign, the error in truncating either series is less than the magnitude of its last term.

For the case $m=0$, for which $\rho_m(x)$ is the largest, a semi-convergent series, suitable for the rapid numerical evaluation of $\rho_o(x)$ when $k_oc \gg 1$, can be derived as follows:

$$\rho_o(x) = \frac{1}{2\pi} \; e^{k_oc} \int_0^\infty e^{\frac{x}{c}\alpha} \frac{(\alpha \sin \alpha + k_oc \cos \alpha)}{\alpha^2 + k_o^2 c^2} \, d\alpha = \frac{e^{k_oc}}{2\pi} \, \mathrm{Im} \int_0^\infty \frac{e^{\frac{\alpha}{c}(x+ic)}}{\alpha - ik_oc} \, d\alpha \tag{27}$$

where Im denotes "the imaginary part of." Integrating successively by parts yields

$$\int_0^\infty \frac{e^{\alpha(\frac{x}{c}+i)}}{\alpha - ik_oc} \, d\alpha = \frac{1}{(ik_o)(x+ic)} + \frac{c}{x+ic} \int_0^\infty \frac{e^{\frac{\alpha}{c}(x+ic)}}{(\alpha - ik_oc)^2} \, d\alpha$$

$$= \frac{1}{(ik_o)(x+ic)} - \frac{1}{(ik_o)^2(x+ic)^2} + \frac{2!c^2}{(x+ic)^2} \int_0^\infty \frac{e^{\frac{\alpha}{c}(x+ic)}}{(\alpha - ik_oc)^3} \, d\alpha$$

$$= \frac{1}{(ik_o)(x+ic)} - \frac{1}{(ik_o)^2(x+ic)^2} + \ldots + \frac{(-1)^{n-1}(n-1)!}{(ik_o)^n(x+ic)^n} + R_n \tag{28}$$

where R_n is the remainder

$$R_n = \frac{n!c^n}{(x+ic)^n} \int_0^\infty \frac{e^{\frac{x}{c}(x+ic)}}{(\alpha - ik_oc)^{n+1}} \, d\alpha$$

But

$$|R_n| < \frac{n!c^n}{(x^2+c^2)^{n/2}} \int_0^\infty \frac{e^{\frac{x}{c}\alpha} \, d\alpha}{(\alpha^2+k^2c^2)^{\frac{n+1}{2}}} < \frac{n!c^n}{(x^2+c^2)^{n/2}(k_oc)^n} \int_0^\infty e^{\frac{x}{c}\alpha} \, d\alpha$$

or

$$|R_n| < \frac{n! \; c^{n+1}}{x(x^2+c^2)^{n/2}(k_oc)^n} \tag{29}$$

For large vaues of k_oc this upper bound for R_n will diminish with increasing values of n to some minimum value, and then increase for larger values of n. Thus the series (28) will be useful for values of k_oc for which the minimum value of the remainder is sufficiently small. Taking the imaginary part of (28), we obtain

$$\rho_o(x) = -\frac{x\, e^{k_oc}}{2\pi k_o(x^2+c^2)}\left[1 + \frac{2c}{k_o(x^2+c^2)} - \frac{2(x^2-3c^2)}{k_o^2(x^2+c^2)^2} - \frac{24c(x^2-c^2)}{k_o^3(x^2+c^2)^3}\right.$$

$$\left. + \frac{24(x^4-10c^2x^2+5c^4)}{k_o^4(x^2+c^2)^4} + \frac{240c(3x^4-10c^2x^2+3x^4)}{k_o^5(x^2+c^2)^5} + \ldots\right] \qquad (30)$$

A convergent series for $\rho_o(x)$, suitable for its numerical evaluation when $k_oc \ll 1$, can be derived on the basis of the formulas[7]

$$\left.\begin{aligned} &\int_0^\infty \frac{e^{-\lambda\alpha}d\alpha}{\alpha^2+a^2} = \frac{Z}{a}, \qquad Z = \mathrm{Ci}\, X \sin X + (\tfrac{\pi}{2} - \mathrm{Si} X)\cos X \\ &\int_0 \frac{e^{-\lambda a}\alpha^{2n}d\alpha}{\alpha^2+a^2} = (-1)^n a^{2n-1} Z + \sum_{r=1}^{n} (-1)^{r-1} \frac{(2n-2r)!\, a^{2r-2}}{\lambda^{2n-2r+1}} \end{aligned}\right\} \qquad (31)$$

where

$$X = \lambda a$$

$$\mathrm{Si}\, X = \int_0^X \frac{\sin x}{x}\,dx, \qquad \mathrm{Ci}\, X = \int_\infty^X \frac{\cos x}{x}\,dx$$

Then putting $\lambda = -x/c$ and $a = k_oc$, and introducing the series expansion of $\sin\alpha$ and $\cos\alpha$ in (27), we obtain

$$2\pi\, \rho_o(x) = e^a \left\{ a \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \int_0^\infty \frac{e^{-\lambda\alpha}\alpha^{2n}}{\alpha^2+a^2}\,d\alpha + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!} \int_0^\infty \frac{e^{-\lambda\alpha}\alpha^{2n}}{\alpha^2+a^2}\,d\alpha \right\}$$

or, from (31),

$$,(x) = e^{a}\left\{Z\left[\sum_{n=0}^{\infty}\frac{a^{2n}}{(2n)!}-\sum_{n=1}^{\infty}\frac{a^{2n-1}}{(2n-1)!}\right]+\sum_{n=1}^{\infty}\sum_{r=1}^{n}\frac{(-1)^{n-r+1}(2n-2r)!a^{2r-1}}{(2n)!\,\lambda^{2n-2r+1}}\right.$$

$$\left.+\sum_{n=1}^{\infty}\sum_{r=1}^{n}\frac{(-1)^{n-r}(2n-2r)!a^{2r-2}}{(2n-1)!\lambda^{2n-2r+1}}\right\}$$

$$= Z + e^{a}\left[\sum_{r=1}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^{s+1}(2s)!a^{2r-1}}{(2s+2r)!\,\lambda^{2s+1}}+\sum_{r=1}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^{s}(2s)!a^{2r-2}}{(2s+2r-1)!\lambda^{2s+1}}\right]$$

$$= Z + e^{a}\sum_{s=0}^{\infty}\frac{(-1)^{s}}{\lambda^{2s+1}}\left[\frac{1}{2s+1}-\frac{a}{(2s+1)(2s+2)}-\frac{a^{2}}{(2s+1)(2s+2)(2s+3)}\cdots\right] \tag{32}$$

we have[8]

$$e^{a}\sum_{r=1}^{\infty}\frac{(-1)^{r-1}a^{r-1}}{(2s+1)(2s+2)\ldots(2s+r)} = \sum_{r=1}^{\infty}\frac{a^{r-1}}{(r-1)!(2s+r)} \tag{33}$$

tituting (33) into (32) yields

$$2\pi\rho_{o}(x) = Z + \sum_{s=0}^{\infty}\sum_{r=1}^{\infty}\frac{(-1)^{s}a^{r-1}}{(r-1)!(2s+r)\lambda^{2s+1}} \tag{34}$$

we also have

$$\sum_{s=0}^{\infty}\frac{(-1)^{s}}{(2s+1)\lambda^{2s+1}} = \cot^{-1}\lambda$$

$$\sum_{s=0}^{\infty} \frac{(-1)^s}{(2s+1)\ \lambda^{2s+1}} = \frac{\lambda}{2} \ln \left(1 + \frac{1}{\lambda^2}\right)$$

$$\sum_{s=0}^{\infty} \frac{(-1)^s}{(2s+3)\ \lambda^{2s+1}} = -\ \lambda^2 \left(\cot^{-1} \lambda - \frac{1}{\lambda}\right)$$

$$\sum_{s=0}^{\infty} \frac{(-1)^s}{(2s+4)\ \lambda^{2s+1}} = \frac{\lambda^3}{2} \left[\ln \left(1 + \frac{1}{\lambda^2}\right) - \frac{1}{\lambda^2}\right], \text{ etc.}$$

Substituting these series into (34) and collecting the coefficients of $\cot^{-1}\lambda$ and $\ln(1 + 1/\lambda)$, we obtain finally

$$2\pi\rho_o(x) = \left[\mathrm{Ci}X + \frac{1}{2} \ln\left(1 + \frac{1}{\lambda^2}\right)\right] \sin X + \left(\frac{\pi}{2} + \cot^{-1}\lambda - \mathrm{Si}X\right) \cos X$$

$$+ \frac{X^2}{2!\lambda} - \left(\frac{1}{\lambda} - \frac{1}{3\lambda^3}\right) \frac{X^4}{4!} + \left(\frac{1}{\lambda} - \frac{1}{3\lambda^3} + \frac{1}{5\lambda^5}\right) \frac{X^6}{6!} - \cdots$$

$$+ \frac{1}{2\lambda^2} \frac{X^3}{3!} - \left(\frac{1}{2\lambda^2} - \frac{1}{4\lambda^4}\right) \frac{X^5}{5!} + \left(\frac{1}{2\lambda^2} - \frac{1}{4\lambda^4} + \frac{1}{6\lambda^6}\right) \frac{X^7}{7!} - \cdots \qquad (35)$$

where

$$X = \lambda a = -k_o x\ ; \qquad \lambda = -x/c\ , \qquad a = k_o c$$

For large values of λ, (32) yields the asymptotic formula

$$2\pi\rho_o(x) \approx \mathrm{Ci}\ X \sin X + \left(\frac{\pi}{2} - \mathrm{Si}\ X\right) \cos X$$

$$+ \frac{e^a - 1}{\lambda a} - \frac{e^a}{3\lambda^3} \left(1 - \frac{a}{4} + \frac{a^2}{4\cdot 5} - \frac{a^3}{4\cdot 5\cdot 6} + \cdots\right) \qquad (36)$$

which is more convenient than (35) for computing values of $\rho_o(x)$ when $\lambda > 5$ and $a < 1$.

Values of $\rho_m(x)$ and of the amplitudes of the far-field terms in (23),

$$\zeta_{mf} = 4\pi \frac{k_o + k_m}{k_m} e^{-\frac{c}{2}(k_o+k_m)}$$

are given in the following table for the case $b/c = 10$ and $\frac{U}{\sqrt{bg}} = 1.0$.

$\rho_m(x)$

		$\lambda = -x/c$										
m	ζ_{mf}	1.0	2.0	3.0	5.0	7.0	10	15	20	50	70	100
0	22.74	.3335	.2584	.2167	.1700	.1422	.1159	.0886	.0719	.0335	.0244	.0173
1	6.87	.0437	.0122	.0043	.0007	.0001	0					
2	3.53	.0261	.0046	.0009	.0001	0						
3	1.86	.0203	.0021	.0002	0							

It is seen that the terms corresponding to $m = 0$ are the principal contributors to the surface disturbance. Indeed, the near-field terms for $m > 0$ are sufficiently small so that they could reasonably be neglected after a very short distance downstream from a source. This is not the case, however, for $m = 0$, for which the near-field terms remain appreciable for distances downstream equal to many multiples of the depth. Hence, if, as was stated earlier, one must take into account a source distribution within the wake in determining the wavemaking of a ship form in a real fluid, the neglect of the near-field part of the velocity potential under these conditions seems even less permissible.

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DISCUSSION

by H. Maruo

To determine the wave resistance by the measurement of the wave profile has now become a fashion and a considerable number of papers about it are presented at this seminar. Contrary to the optimisn exposed by others, Professor Landweber's critical contribution expresses a pessimism on the possibility of this method rather than its accuracy. The method of wave analysis is possible only when the asymptotic expression can give, with sufficient accuracy, the wave profile and more rapidly decaying local term has no contribution in effect within the accuracy of the experiment. There is some doubt especially when the analysis is made along a longitudinal cut. According to the author's result, the wave analysis is useless because the measurement includes the wave profile at any value of x not sufficiently large. Professor Landweber has derived his conclusion from the computation of ρ_o. It is of some interest to observe that ρ_o gives the case of a two-dimensional source extending infinitely along a line perpendicular to the stream. So it has no decay in the direction of y. Usually the wave profile is discussed without regarding the tank wall. In the case of infinite stretch of water surface, the fluid motion is quite three-dimensional and we can expect a rapid decay of the local or so-called near-field term. It looks rather strange that the velocity potential and the surface elevation have completely different mathematical expressions bstween the case of unbounded water and that with side wall even if the width is very large. The Fourier integral of continuous spectrum in the former case turns to the Fourier series with discrete spectrum as soon as the side wall is introduced. Even in a big ocean the water is not infinite, so that the series expression may have to be preferred to the integral representation. Such a mystery should be dissolved by the fact that the Fourier integral is a limit of the Fourier series. In fact, Mitchell, himself, obtained his well-known result from the limit of Fourier series. Therefore, the series representation will become identical to the integral representation for infinite width when the width of the channel becomes sufficiently large. Since the function ρ_o is independent of the width b, Landweber's discussion about it is valid also in the case of infinite width. In the latter case, each term of the series degenerates to the integrand within a small interval of the integration variable. The integral representation when the width is infinitely large is given by the Equation (6) of this paper and ρ_o corresponds to the integrand at $\theta = 0$ and not the integral itself! His result is not so surprising because it is not for the three-dimensional flow but for the two-dimensional flow. He emphasizes the effect of the source distribution which represents the wake. However, his conclusion is not the consequence of such a source distribution at all but is due to the mehtod by which the conclusion is derived. Notwithstanding these points, Professor Landweber's critical essay must be appreciated because so little attention has been paid to the magnitude of the local disturbance in the wave analysis.

by Dr. Gadd

I should like to ask Professor Landweber a brief question concerning his paper. It is this: - Are the strengths of the wake sources on the Betz-Tulin approach proportional to the local rate of change of the wake displacement area? If this is so, then since downstream of the model displacement area $\rightarrow$ momentum area $\rightarrow$ approximately constant, the wake source strength should quite rapidly become zero downstream. Thus providing the wave measurements are not made too close to the model, no serious errors should arise in the estimation of C_w .

by Dr. Hogben

I would like to add a footnote to Dr. Gadd's remarks about the downstream extent of the wake source distribution. It is useful for this purpose to consider the variable x/c (in the table on page 14 of Professor Landweber's paper) as a multiple of transverse wavelength λ_0 . For the case quoted I find that $x/\lambda_0 = (0.05/\Pi)(x/c)$ so that when $x/c = 20$ for example $x/\lambda_0 \doteq 1/\Pi$ and $\rho(0) = 0.0719$. Thus if it is true, as Dr. Gadd has suggested, that the wake source distribution is localized near the stern of the model then already 1/3 of a transverse wavelength downstream the effect of the near field is small in relation to the order of discrepancy which is to be explained. I therefore feel that Professor Landweber is overestimating the importance of this wake source effect as a possible cause of serious error in the analysis of the wave pattern measurements.

by K. Eggers

To some amount my reflections to Dr Landweber are identical with arguments of Mr. Sharma and Prof. Maruo, presented with much more eloquency than I could have produced. I appreciate this paper as a pendant to my paper submitted, where I derive velocity potential of source in a canal as well in order to apply Green's theorem, for investigation of local wave decay behind any kind of disturbance in a canal, where the free wave system is shown not to die out at all, as it is an almost periodic function. I am impressed by the numerical behavior of functions $\zeta_1(x)$ and $\zeta_2(x)$, but I trust to agree with Dr. Landweber that these ratios are not typical for the local-field decay of components due to a body in ideal fluid but only for sources representing the wake, and that relative importance of the two or three fiest terms will fade off with increasing tank width.

by S. D. Sharma

In the introduction to his very interesting paper Dr. Landweber advocates a rather wholesale rejection of 'the results obtained by the method of surface profile measurements'. He then proceeds to propose, and consider in detail, two possible causes of error, 'one physical and the other mathematical'. Although the author's criticism refers to previous published data, I think it may be useful to review my own wave-analysis results presented at this Seminar in the light of his conclusions.

The author's dissatisfaction with the results of wave analysis arises mainly from the fact that the resistance components obtained from wake and wave analyses in general do not add up to the total measured resistance of the model. In this connection it should be remembered that neither the viscous nor the wave resistance can be rigidly defined for a ship model moving on the free surface of a real fluid. Rigorous definitions are, however, possible in special, idealized cases. Thus there are exact definitions for the wave resistance in a perfect fluid, and for the viscous resistance of a deeply submerged double model. Now, useful - although non-rigorous - definitions can be derived for the viscous and wave resistance of a model in a real fluid by analogy with the special cases just mentioned. For example, the typical mechanism of momentum and energy transfer in the viscous wake or wave flow can be used to define a certain viscous and wave resistance in a real fluid. Thus Equation (2) of the author's paper, if applied to the actual wake of a ship model, amounts

to a formula for calculating a phantom viscous resistance by effectively neglecting the presence of the free surface. Similarly, the proposed methods for the calculation of wave resistance from measured surface profiles are based on the assumption that the actual free-surface deformation can be interpreted and analysed in the same way as if we were dealing with a perfect fluid. If non-rigorous viscous and wave resistance are thus defined and evaluated independently of each other, in my opinion there is no reason a priori why they should add up exactly to the total measured resistance. In fact it is obvious that an interference term of some sort is needed in a real fluid to account for the interaction between viscous and wave phenomena. Moreover, it should be noted that in the experiments discussed by the author the analogy definitions indicated above had to be further simplified by neglecting certain higher order terms. Hence the discrepancy pointed out by the author does not necessarily mean that the wave-analysis methods are 'seriously in error'.

Let us now briefly consider the author's proposals concerning the possible causes of error in the results of wave analysis. Firstly, he refers to a difficulty inherent in the analogy method of defining wave resistance in a real fluid, namely the extrapolation of potential flow into the viscous wake. On the basis of my recent experience this difficulty seems to me to be less serious than it appears at first sight. If the wave analysis is based on the concept of the linearized free-wave system, the potential can be derived from a few surface profiles, e.g. from two transverse sections. The problem therefore reduces to an analytical continuation of the transverse section across the viscous wake. It is true that, theoretically, two complete transverse sections without any interruptions are necessary for the analysis. In practice, however, it was observed that if a sufficient amount of redundant information is available, i.e. if instead of two, say five or more sections are simultaneously analysed by taking an average according to the method of least squares, then the final result is comparatively insensitive to small changes over limited regions of the individual sections. Thus the distorting influence of the questionable part of the measured transverse section across the viscous wake can be made sufficiently small by increasing the number of sections to be measured for the analysis. In fact, in case of the stereophotogrammetrical measurements the part of the transverse section across the viscous wake could not be measured at all for technical reasons and therefore had to be arbitrarily interpolated. Yet the results obtained from a detailed analysis of 31 sections at one speed are not only self-consistent but also in agreement with the results of an independent analysis of measurements obtained by a sonic transducer at the same speed as can be verified on reference to my paper to this Seminar (Figure 5).

The author's second objection refers to the use of the linearized free-wave system for representing the actual free-surface deformation behind a ship model, i.e. essentially neglecting the so-called local wave. The author rightly implies that the wave flow generated by a model in a viscous fluid outside the wake at a great distance from the model is more nearly represented by a discrete source rather than a dipole - as in a perfect fluid. Since the local wave of a source decays less rapidly than that of a dipole the objection raised by the author seems to be particularly pertinent in a real fluid, all the more so because sources may have to be situated in the wake behind the model to generate the outside flow. In any actual wave analysis the presence of a small local wave - like any other distortion of the free-wave system - can be essily detected by plotting the Fourier coefficients of the transverse sections over the distance behind the model. Only the free-wave components will yield sine waves of the prescribed wave length (see Figure 4 of my paper). All other components will in general be non-oscillating terms. The local-wave component will probably be a rapidly decaying monotonous term. By assuming a suitable polynomial approximation it can be easily filtered out to yield the pure free-wave component. In our detailed investigation of the waves generated by an Inuid in 15 different speeds no clear evidence of a sensible local wave was found in the region analysed. Of course, for another model and a different speed range the local wave may be quite prominent. However, the important point is that all small distortions of the free-wave system (including local waves) can be detected and eliminated if a sufficiently large number of sections is used for the analysis.

Summing up, the experimental evidence available up to date does not suggest that the discrepancy puzzling the author - and indeed everyone of us engaged in wave analysis - can be satisfactorily explained by the arguments advanced in this paper.

AUTHOR'S REPLY

The slow reduction in magnitude of ρ_0 with downstream distance from the source has elicited comment from several of the discussors. Before replying to each of them, it will be helpful to note that ρ_0 depends only upon $|x_0/c|$ and k_0c ; it is independent of b/c , as has been observed by Dr. Maruo. Thus the values of ρ_0 in the given table, corresponding to $m = 0$, may be interpreted for values other than $b/c = 10$. This leads to a criteron for the distance downstream from a source at which the near field term becomes negligible, if only the magnitude of ρ_0 is considered. Selecting from the table the value $x/c = 100$ as the requisite downstream distance in terms of source depth, we have

$$\frac{x_0}{b} = \frac{100c}{b}$$

as the corresponding distance in terms of tank width. But also, since the table has been computed for the case $k_0c = 0.10$, we obtain for the associated Froude numbers,

$$F_b = \frac{U}{gb} = \frac{1}{k_0b} = \frac{0.1x}{b}$$

or

$$\frac{x_0}{b} = 10\ F_b^2$$

Hence, at the reasonable Froude number $F_b = 0.31$, we have $x_0/b = 1$, i.e. the near-field disturbance of the source becomes negligible after a downstream distance equal to the width of the tank.

Dr. Gadd has raised a most interesting point. It is true that the expression defining the strength of the Betz source distribution resembles that for the rate of change of the wake displacement area. These differ only in that the velocity distribution of an analytically continued potential flow is used in defining the source strength instead of the single velocity outside the wake used to define the displacement area. This difference in definition probably becomes unimportant sufficiently far downstream in the wake. Wake data indicate that the displacement area becomes nearly constant after about 6 to 10 diameters of the wake producing object. Dr. Gadd's argument, which is probably valid, then indicates that the strength of the Betz source distribution should become negligible at downstream distances greater than about one model length. But, as was shown above, an additional downstream distance approximately equal to a tank width is required for the near field disturbance to become negligible.

Since model lengths are of the order of magnitude of the tank width, this indicates that the near-field effects of the wake sources may be neglected beyond a downstream distance of about two model lengths.

Dr. Hogben points our that a downstream distance of 20 source depths is equal to only about 1/3 of the transverse wave length associated with $m = 0$. This in true, but it seems to be more meaningful to express this distance in terms of tank width, or model length. Furthermore, at this downstream distance, the tabulated values show that the near field contribution is still seven percent of the far-field amplitude. It is not until about a distance of 100 source depths downstream that the near field term is reduced to about 1.7 percent of the far-field amplitude.

Dr. Maruo's critical and scholarly evaluation of my contribution is deeply appreciated. I take issue only with his statement that my "conclusion ... is due to the method by which the conclusion is derived," which implies that another method of analysis would have led to a different conclusion. The assumption that the tank width is finite and of the order of magnitude of the model length is a realistic one. Alternative conclusions derived by assuming inginite tank width should not invalidate the results of the present analysis.

Finally, I agree fully with Dr. Eggers remarks. The field of a single source is not a good approximation for the disturbance due to a body in an ideal fluid, for which the total source strength must be zero. The analysis applies mainly to the sources representing the wake in a tank of finite width.

In his provocative discussion Dr. Sharma refuses to accept my "definition" of wave resistance, implying that it gives only one of several terms of his chart of types of resistance which contribute to this component of the drag. My feeling is that such listings of resistance components is rather futile since there is no way of obtaining these terms either experimentally or theoretically, so that I prefer to avoid such classifications. A more useful point of view is that the Betz-Tulin theory yields a rational procedure for separating the viscous drag from the total, and it appears reasonable to refer to the remainder as the "wave drag." I do not consider that this constitutes a definition of wave drag; rather, since the Betz-Tulin theory is not exact, all that should be claimed is that it gives a procedure for obtaining an approximate value for the wave drag. Furthermore, since the theory and the measurements are for the actual flow, including the presence of the free surface, contrary to Dr. Sharma's statement, the effects of a boundary layer and wake on the generated waves are

including in the result, and it appears to me to serve no useful purpose to attempt to analyze the wave resistance further into a part which depends only on the Froude number and another part which depends upon both the Froude and Reynolds numbers.

Secondly, Dr. Sharma states that, in the analysis of his surface profile data, he has already employed the procedure I have suggested of extending the potential flow into the wake. According to his description, because of difficulty in measuring the surface profile across the wake, a linear interpolation was assumed for the surface profile in this region; it is this interpolation which Dr. Sharma is interpreting as the extension of the potential flow into the wake. There are two flaws in this argument. The formula for the wave resistance consists of the sum of two integrals. In one of these the square of the actual surface disturbance, even in the wake region, occurs in the integrand. Hence the procedure described by Dr. Sharma is an approximation imposed by the limitations of the measurements, and not in accord with the requirements of the Betz-Tulin theory. It is the second integral which, in the wake region, involves the components of the analytically continued potential flow. Since this potential-wake flow is associated with a volume distribution of wake sources, it is not obvious that the analytical procedures employed by Dr. Sharma to analyze his surface profile measurements are valid within the wake; and it would be highly fortuitious if it turned out that Dr. Sharma's simple expedient of linear interpolation actually gave the required correction.

DETERMINATION OF THE WAVE RESISTANCE OF A PARTLY IMMERSED AXISYMMETRIC BODY

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DETERMINATION OF THE WAVE RESISTANCE OF A PARTLY IMMERSED AXISYMMETRIC BODY

1. Summary

The total resistance of a slender axisymmetric body was measured in a ship tank by a new method, and the 'residual' resistance was determined. The calculated values of wave resistance for such a model, both according to thin-ship and to slender-body assumptions, are generally very much greater than the observed values, and the theoretical curves of resistance versus speed are more humped. Estimates of the wave resistance have also been obtained from an analysis of the measured wave pattern; they are about three-quarters of the residual resistance over a range of Froude number 0.2 to 0.3.

2. Introduction

This paper presents a comparison between measured values of residual resistance (total drag - (friction + form) drag) and calculated values of wave resistance for an axisymmetric body. The wave resistance has also been determined independently from the measured wave pattern.

Recently, Vossers[1] and Tuck[2] among others have put forward theories which provide expressions for wave resistance on the assumption that a ship form may be regarded as 'slender', that is, having small draught as well as beam in comparison with its length. The equations are not dissimilar from those of thin-ship theory; the experimental tests described were to investigate whether these new theories are in fact an improvement.

The idea of deducing the wave resistance from the actual wave pattern is not new, but until the last few years little experimental work has been done. Eggers[3], Ward[4] and Gadd and Hogben[5] have now published results of investigations, each using different methods of analysis. Results for ship-forms show that wave-making accounts for about one half of the measured residual resistance. Hogben[6] found for a hovercraft (which can be regarded as a moving pressure on the water surface) that wave resistance was about equal to that predicted from theory, but that 125% of the resistance found from force measurements on the hovercraft. The present work seeks to examine this ratio for an axisymmetric body.

3. Experimental Arrangements

Tests were conducted in No. 2 tank of Ship Division, National Physical Laboratory, at Teddington, by kind permission of Mr. A. Silverleaf, the Superintendent. This tank is 20 feet wide and 9 feet deep, the useful run of deep-water being about 300 feet, allowing for acceleration and braking of the carriage. The model was made of Fibreglass (resin-bonded glass fibre) and was of length 12 feet 1 inch and diameter 12 inches. It consisted of three sections: a cusped nose 50 inches long and a similar tail, and a parallel middle-body 45 inches long, (Figure 1).

The model was towed by the carriage at a set speed. The mounting was designed so that trim was prevented, and this was accomplished by suspending the model at each end on pendulums which could swing longitudinally below the carriage. At each end of each pendulum were shafts which ran in transverse bearings, one on the carriage and one on the model. The bearings were air-lubricated, and thus when the pendulums were both vertical, no horizontal force could be transmitted to the model. In order to achieve this, air for the lower bearings was bled off through the upper shafts, (Figure 2).

The total resistance of the model was measured with a 0 - 50 lb. proving-ring dynamometer, fitted with electrical strain-gauges, using a four-arm-active bridge network, whose output was connected to a meter with large time constant. This had the advantage of preventing free longitudinal movement, thus maintaining the pendulums vertical, and also obviating the need to unclamp and clamp the model at the beginning and end of each run. Further, it was not necessary to know the resistance in advance and offset with weights as is usual. The accuracy claimed is about 1%.

The wave-pattern was recorded with depth probes of the capacitance type, mounted on a submerged frame made of 'Dexion' (slotted light-alloy angles), originally developed by Gadd and Hogben[5]. See Figure 3. The outputs of the probes were connected to four channels of an ultra-violet paper recorder. This gave a series of continuous longitudinal sections through the wave-pattern, which moved past the stationary probes.

4. Total and Residual Resistance

The curve of total resistance measured with the proving-ring was reproducible to within 2% over a year, and is shown in Figure 4.

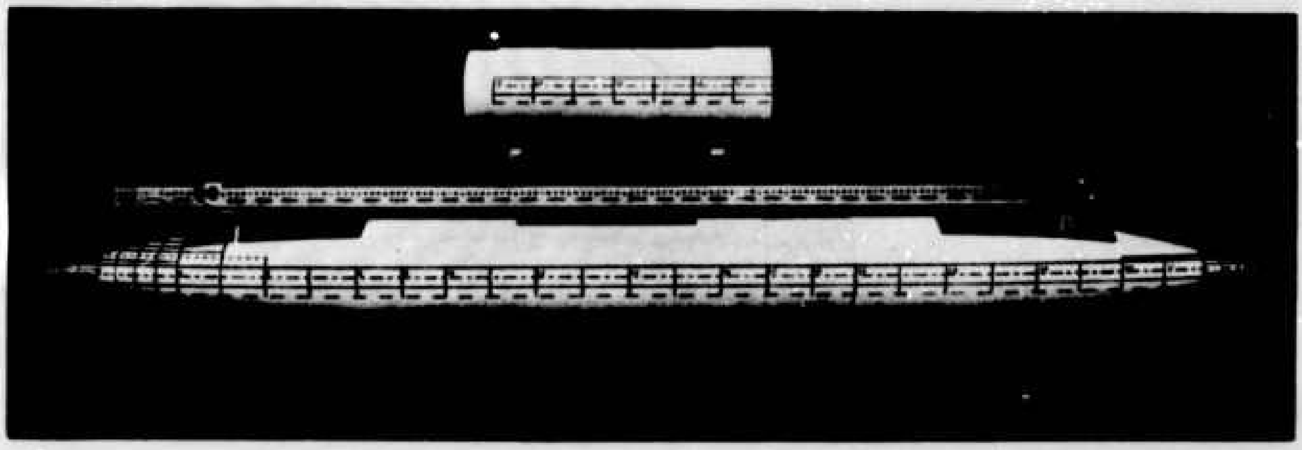


Fig.1.

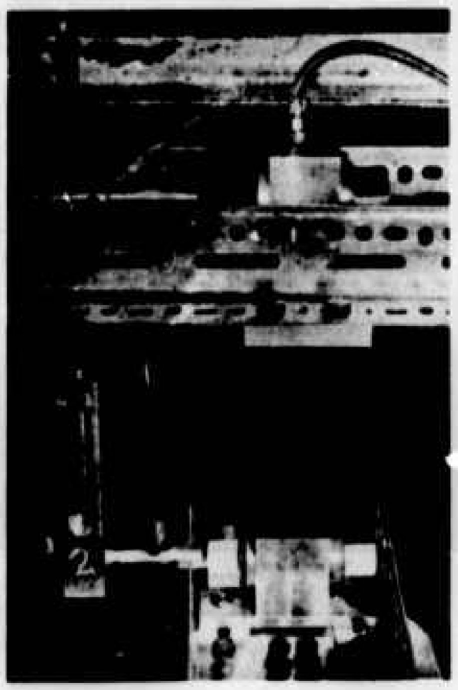


Fig.2.

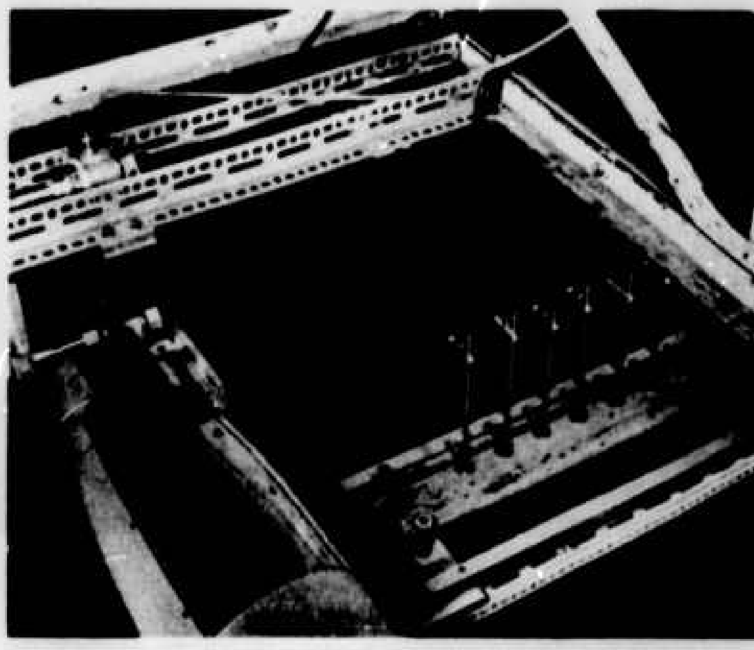
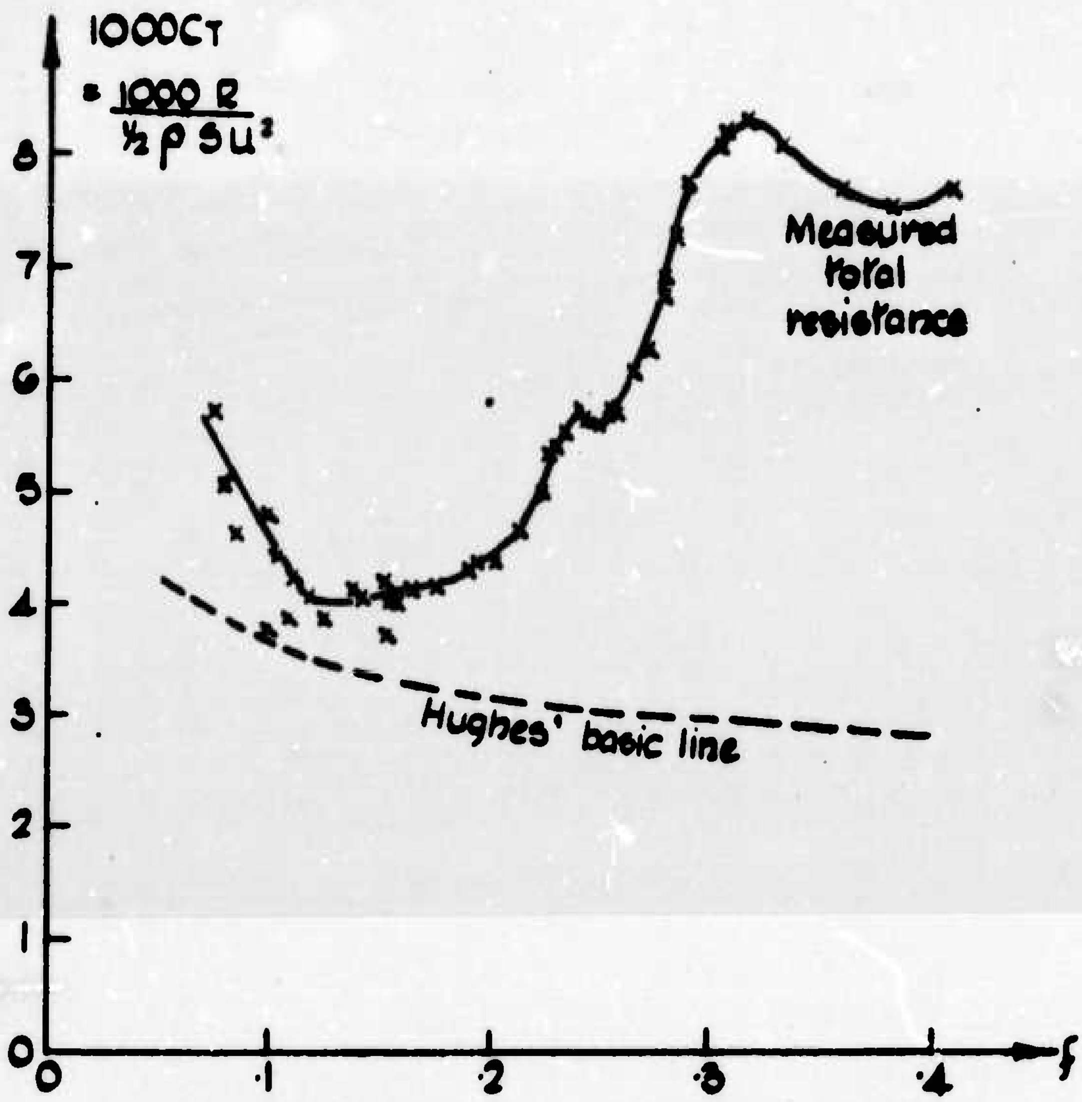


Fig. 3.



TOTAL RESISTANCE CURVE FIG. 4.

Corrections were made, first for the resistance of the two trip wires, then for the blockage due to the finite tank size, and finally for the area of laminar flow. Friction and form resistance were taken to follow the curve of Hughes[7], determined for fully turbulent flow, such that run-in to the curve of total resistance occurred for Froude number of about 0.15 where wave resistance is negligible. This gave a k-factor of 21.3% above the basic line, quite reasonable for a model with block coefficient $C_b = 0.575$. The resultant curve of residual resistance is shown as a solid line in Figure 5.

The theoretical curves derived from Tuck's equation and also, for comparison, from Michell's equation are both shown in Figure 5. About twenty points were considered enough to define the curves; the integration was performed on the EDSAC II computer at Cambridge. Tuch gives for the wave resistance:

$$R = \frac{\rho g^2 \pi l^4}{n^2} \int_0^{\pi/2} (P^2 + Q^2) \sec^3\theta \, d\theta$$

$$P + iQ = \int_{-1}^{1} hh'(x) \exp(ik \sec\theta \, x) \, dx \; .$$

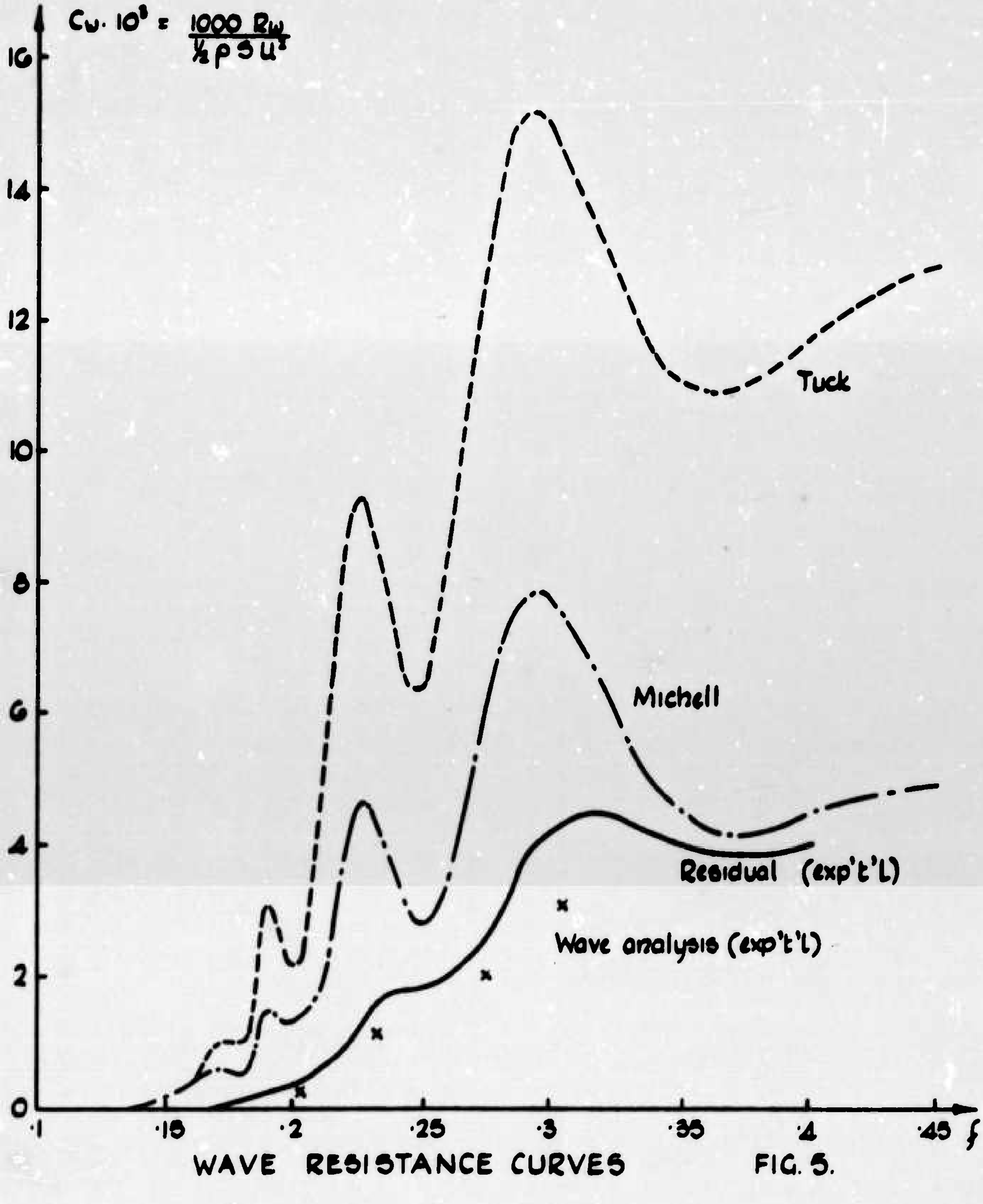
Michell gives (quoted by Wigley[8]) for a body of circular cross-section

$$R = \frac{\rho g^2 \pi l^4}{n^2} \int_0^{\pi/2} (P^2 + Q^2) \sec^3\theta \, d\theta \; .$$

When $\epsilon = r_{max}/l$:

$$P + iQ = \frac{2\epsilon}{\pi} \int_0^{h/\epsilon} \int_{-1}^{1} \frac{hh'(x)}{\sqrt{(h^2 - \epsilon^2 z^2)}} \exp(-\epsilon k \sec^2\theta z + ik\sec\theta x) \, dx dz,$$

where x = dimensionless length, -1 at bow, +1 at stern
$r(x)$ = radius at given section
l = half total length
$h(x) = r(x)/l$
$k = 1/2f = gl/u^2$
u = axial velocity .



WAVE RESISTANCE CURVES FIG. 5.

The curves of Figure 5 were obtained by numerical integration, and it is clear that their general shapes are the same, although Tuck's estimate is roughly twice Michell's. Both curves show the usual humps and hollows, and these may be seen to occur at Froude numbers about 6% below those for the experimental curve.

The latter effect is frequently observed, e.g., by Wigley[9], and seems to be a displacement effect of the boundary layer. This general cause is also believed to be responsible for the flattening of the humps and hollows (Havelock[10]). It is hoped to carry out more work on these two effects next year.

However, the most noticeable feature of the results is the difference in the magnitudes. Some reduction in resistance may occur because the model is not vertically-sided above the mean water level, but this would seem most unlikely to be the 75% necessary to bring the experimental curve up to the level of Tuck's theoretical one. One is accordingly led to the conclusion that there exists a fundamental discrepancy between theory and experiment.

5. Wave Pattern Analysis

The method used here follows that put forward by Gadd and Hogben[5]. The principle is to consider the energy in the wave-pattern in an area bounded by two transverse lines behind the model. For a succession of longitudinal strips, corresponding to the wave-probes, dR_w/dy is calculated from the Fourier components of the longitudinal wave traces, and R_w is obtained by integrating across the width of the tank.

The wave-records were made at four speeds, $u = 4.0, 4.6, 5.4, 6.0$ feet per second ($f = 0.203, .233, .275, .305$). The wave-height was read off every 0.05 second, and punched on 5-hole paper tape for analysis on the EDSAC II computer. Analysis was performed for Fourier ranges of one, two and three transverse wavelengths (for the two-dimensional long waves behind the model the wavelength $\lambda_o = 2\pi u^2/g$) in the area between the stern point of the model and $4\lambda_o$ further downstream. It yielded values of dR_w/dy at each probe, tabulated in columns corresponding to the various Fourier ranges.

The wave-comb frame, (Figure 3), held probes at 3 inch intervals from 18 inch to 72 inch from the center-line. Two further positions were used, at 93 inch and 114 inch from the center-line; probes for these are attached to the tank wall.

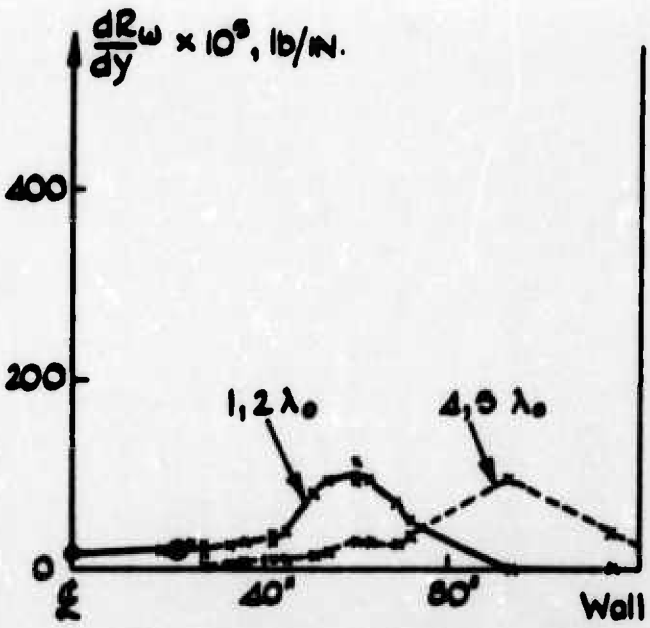
Typical results at each of the four speeds are shown in Figures 6 through 9. Some of the points were repeated, as may be seen; over half were within 5%, i.e., the wave height was reproduced within 2-1/2%.

A further check was made by measuring the wave-form on a longitudinal section with point gauges suspended from a boom attached to the carriage. At first this was carried out on the center-line only, and values of dR_w/dy less than expected by 30 - 50% were found, at all speeds, and for several wavelengths downstream. But repeating the check at 21 inches from the center-line gave good agreement (about +4%), with only one serious discrepancy.

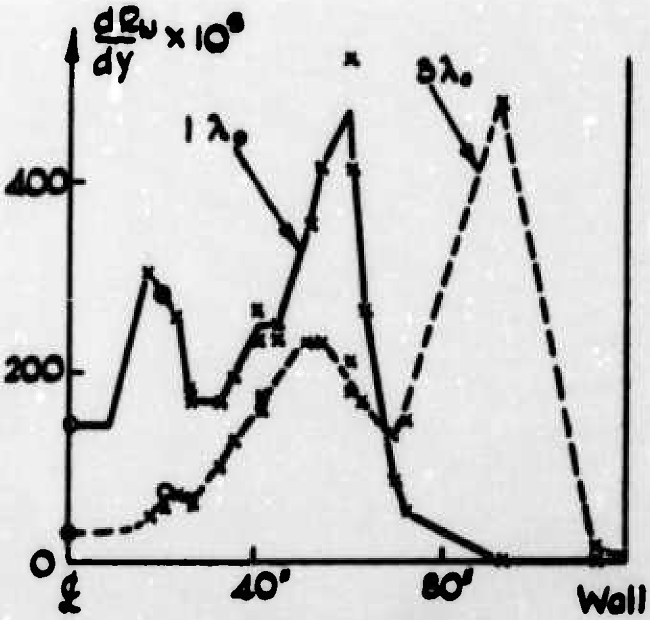
The areas under the curves were averaged, assuming that they were unbiassed estimates of the resistance, which seemed reasonable as the scatter was quite small (average standard deviation = 6.1%). In the table opposite, the limits on the final value of $10^3 C_w$ are shown as one standard deviation of the mean each way.

Range	u = 4.0	u = 4.6	u = 5.4	u = 6.0
1 λ_o	0.855	[illegible]53	11.43	20.87
2	0.934	4.73	10.80	22.62
3	0.947	5.15	12.67	21.27
4	1.051	4.24	10.94	(13.95)
1,2 λ_o	0.838	4.80	10.95	22.71
2,3	0.841	4.86	12.79	21.91
3,4	0.967	4.88	13.26	(18.34)
1,2,3 λ_o	0.908	5.03	11.86	23.73
2,3,4	0.967	4.87	11.86	20.61
Mean	0.923	4.79	11.84	21.96
R_w, lbs.	0.074	0.383	0.947	1.758
$10^3 C_w$	0.296 $\pm .007$	1.15 $\pm .02$	2.06 $\pm .05$	3.11 $\pm .06$

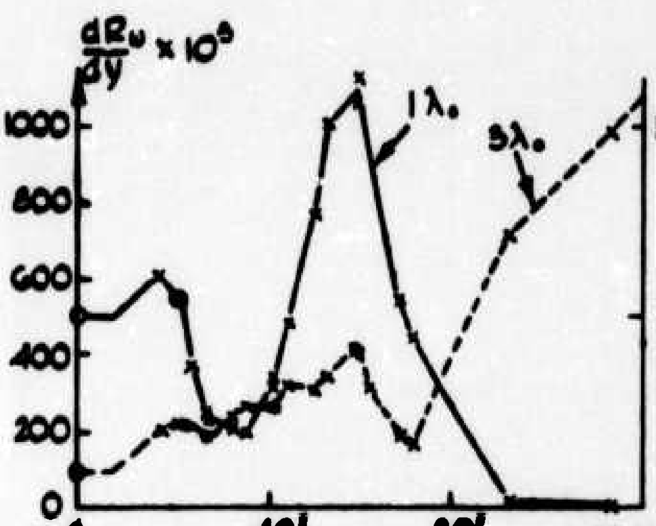
(Areas are in square inches, taken from the original graphs).



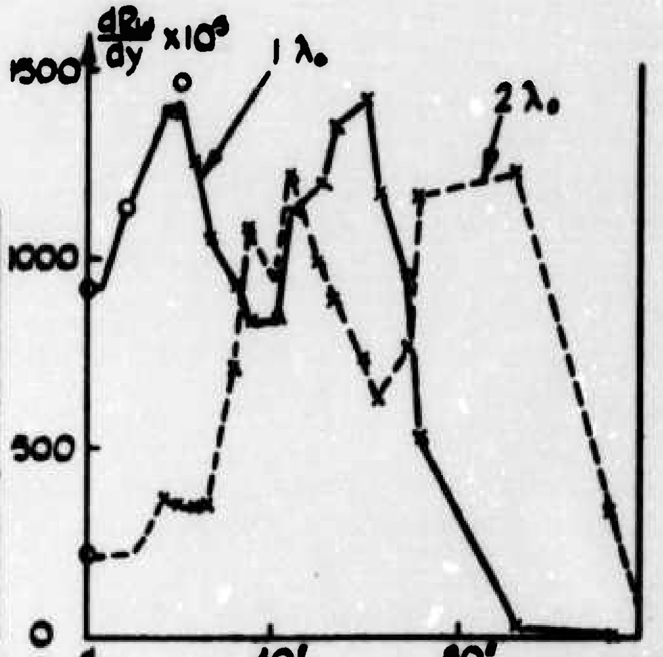
U = 4.0 FIG. 6.



U = 4.6 FIG. 7.



U = 5.4 FIG. 8



U = 6.0 FIG. 9

6. Discussion of Results

The values of the coefficient of residual resistance, with which the above figures should be compared, are seen from Figure 5 to be

0.40 1.57 2.64 4.35

The ratios of the measured wave resistances to the residual resistances are therefore

0.741 0.734 0.779 0.715

Now if the wave resistance could be more closely related to the theoretical wave resistance, than to the residual resistance, one would expect that the second and fourth of these ratios would be higher than the other two, as they refer to speeds which give maxima on the theoretical C_w curve. In fact the reverse is true, and as the actual variation is quite small (s.d. = 3.1%), it therefore appears that for the range of speed considered, $0.2 < f < 0.3$, wave-making was responsible for a constant proportion of residual resistance, 74%.

It has been suggested that in making an estimate of wave resistance in this way, the ranges considered should not approach too closely to the model because of local effects. There is some evidence for this in the above table. For the one λ_o range, the area is always about 5% too low, and a maximum is reached at $4\lambda_o$(u = 4.0), $3\lambda_o$(u = 4.6, 5.4), $2\lambda_o$(u = 6.0), i.e., at a roughly constant distance of 130 inches downstream from the stern-point. No reflexion has occurred and the bow wave, stern wave and transverse wave are each well separated. But it is quite possible that the real explanation lies in the distribution of probes, which are very sparse towards the wall. If there had been enough to have been spread evenly right across the tank, the effect might not have been noticed.

The effect of reflexion at the wall may be estimated by considering ranges somewhat further downstream. The traces at one speed, u = 5.4 feet per second were analyzed over wavelengths 9 - 12, and gave areas under the dR_w/dy versus y curve as follows:

9 λ_o	11.01	9,10 λ_o	10.71
10	9.01	10,11	10.31
11	9.09	11,12	11.16
12	10.52		
		Mean	10.26

The standard deviation of the difference of the means is $\sqrt{(sd_1^2/n_1 + sd_2^2/n_2)} = 0.415$ the difference of the means is $11.84 - 10.26 = 1.58$ which equals nearly four standard deviations, and is statistically very highly significant. We therefore conclude that reflexion decreases the measurable wave energy, here by about 13%.

The effect of the length of range chosen for analysis seems small. Careful comparison suggest an increase in area under the curve of about 5% as analysis is carried out over more than one transverse wavelength, but the scatter is such that this difference is not significant. It is not therefore established, at least by the present work.

The right-hand maxima on all the curves of Figures 6 through 9 represent the energy contained in the bow wave, the second hump behind, visible in Figures 7 through 9, represents the stern wave. It may be noticed that this hump is rather more pronounced in Figures 7 and 9, which correspond to maxima on the R_w curve, than in Figure 8, which does not. This is to be expected on physical grounds, as at wave-resistance maxima, the bow wave reinforces the stern wave, while at minima, it cancels it. The positions of the bow and stern maxima at $u = 5.4$ ft./sec are plotted in Figure 10, which illustrates the wave pattern.

The direct effect of the viscous wake is quite small - about 10% close to the model and much less downstream (compare $1\lambda_o$ and $3\lambda_o$ curves in Figure 7). The discrepancy between residual resistance and wave resistance is thus only made up in part; further investigation will be needed to explain the difference that remains.

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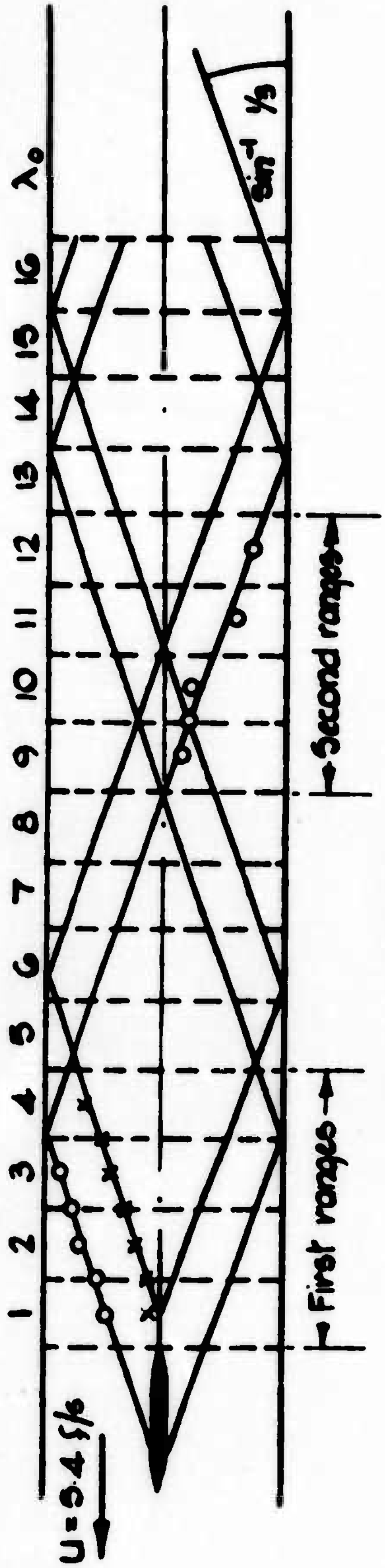

WAVE PATTERN SCALE 1:200

FIG. 10

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A FOURIER TRANSFORM METHOD FOR CALCULATING WAVE-MAKING RESISTANCE FROM WAVE HEIGHT ON A LINE PARALLEL TO A SHIP'S TRACK

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ABSTRACT

It is theoretically possible to calculate wave-making resistance if the wave height is known at all points on a line parallel to the track of a ship which has been traveling for a long time in a straight line on an infinitely wide, infinitely deep body of water. A simple formula for the wave-making resistance is derived employing an exponential Fourier transform of the wave height over a semi-infinite segment of the line parallel to the ship's track. Although ship speed could also in theory be derived from this information, in this paper it is assumed to be known.

When the errors of the technique are examined, the most troublesome are found to be those caused by using the derived formula over a finite rather than a semi-infinite segment of the line. However, if the general shape of the curve of amplitude of the elementary waves as a function of direction is known, these particular errors can be estimated.

I. Introduction

Efforts have been made recently by several investigators to find a way to measure the wave-making resistance of a ship from the waves themselves. These efforts are motivated particularly by the fact that the system used for many years at model basins contains several large unknowns in its correction factors. That method consists of measuring the total resistance of the model and then subtracting the resistance attributed to other factors than wave-making, but the amount of the resistance caused by these other factors is not known precisely. As a result it is questionable whether the resistance which is ascribed to wave-making is really all caused by waves.

This paper shows that the wave height measured at all points on a line parallel to the path of a ship traveling in open water provides enough information to calculate the wave-making resistance. The necessary formulae for calculating it are developed as part of the proof of the proposition. The significance of this finding is that the wave-making resistance of either a mosel or a ship can be found by measuring the wave height continuously as a function of time at one point while the ship or model runs past the point at constant speed. This provides not only a way to find the wave-making resistance of models, but a simple way to measure the wave-making resistance of a full-scale ship. It has the advantage that it does not involve any other resistance than that caused by the waves. It appears to have the disadvantage of requiring a very long and wide towing tank to obtain accurate results.

II. Derivation of a Formula for Calculating Resistance From Observations on a Semi-Infinite Line Parallel to the Ship's Track

It is well known that if a ship is traveling in a straight line at a constant speed on an infinitely deep, infinitely wide ocean, it produces a wave pattern which in the vicinity of the ship assumes a constant shape when described in coordinates moving with the ship. This wave pattern can be described by superposing a set of "elementary waves", each of constant direction and wave length, and of the same amplitude everywhere behind the wave front through the disturbance which creates it. If the speed of the ship is constant, then the wave length of each elementary wave is a function only of the direction in which it moves, and there is only one wave length for each direction. There are two directions for each wave length, and they are symmetrical on either side of the direction of motion of the ship. It can also be shown that the distance between crests of these elementary waves when measured along a line parallel to the line of ship motion is a unique function of the absolute value of the angle between the direction of motion of the ship and the direction of motion of the wave. These considerations lead to the conclusion that it must be possible to find the amplitudes of the elementary waves if enough is known about the wave height on a line parallel to the track.

1. Statement of the Theorem and its Corollary

Theorem:

If the height of the waves produced by a ship is known at each point along a straight line parallel to the path of travel of the ship, and the distance between the ship's track and this straight line is known, then it is possible to decompose the entire wave pattern into its elementary waves, obtaining the amplitude of each wave as a function of its wave lenght and so of its direction of travel.

The wave heights along the line parallel to the ship can, of course, be obtained by a probe which is fixed in one spot and measures the wave height continuously as the ship moves by.

If the amplitude of each elementary wave as a function of its wave length and so of its direction of travel is given, it is possible from known expressions to calculate the wave-making resistance of the ship. We can therefore draw the following additional conclusion:

Corollary:

If the wave height has been measured at one instant at all points along a straight line parallel to the direction of motion of a ship of known speed, and the distance between the ship's track and the straight line is known, then if the ship has been moving in a straight line for a very long time all the information necessary to calculate its wave-making resistance has been obtained.

2. Proof of the Propositions and Development of the Equations for the Resistance

Proof of the theorem:

The waves produced by a ship consist generally of a local disturbance which moves with the ship together with two or more sets of waves which are left behind as the ship moves through the water. Only the waves left behind contribute to the wave-making resistance. An ordinary ship usually leaves behind two distinct trains of waves, one originating at the bow and the other at or near the stern. Sometimes there are more trains of waves, but there is always a fintie number of them and they all originate somewhere between the bow and the stern.

If the wave height corresponding to each of these wave trains is decomposed into its elementary waves, each of which has a single direction and corresponding wave length, then the wave amplitude of each wave train at any point on the surface can be written,[1] if $y > 0$,

$$\zeta = \int_{-\pi/2}^{\tan^{-1}(-x/y)} C(\theta)\cos(x,\cdot)\,d\theta + \int_{-\pi/2}^{\tan^{-1}(-x/y)} S(\theta)\sin(x,y)\,d\theta\,, \qquad (1)$$

where the function (x,y) is defined as

$$(x,y) = K_o \sec^2\theta\,(x\cos\theta + y\sin\theta). \qquad (2)$$

Since the ship is symmetrical in the centerline plane, the wave height is an even function of y and the functions $C(\theta)$ and $S(\theta)$ are even functions of θ . To make Equation (1) apply for $y < 0$ the limits of integration are changed to $\int_{\tan^{-1}(-x/y)}^{\pi/2}$.

Suppose that the hull produces two trains of waves, one originating at the bow where $x = 0$, the other at the stern where $x = -L$, both on the centerline plane. Then the equation of the free waves produced by the hull is simply the sum of the heights of the two trains of waves.

$$\zeta = \int_{-\pi/2}^{\tan^{-1}(-x/y)} C_o(\theta)\cos(x,y)\,d\theta + \int_{-\pi/2}^{\tan^{-1}\cdot(x/y)} S_o(\theta)\sin(x,y)\,d\theta$$

$$+ \int_{-\pi/2}^{\tan^{-1}(\frac{x+L}{-y})} C_L(\theta)\cos(x+L,y)\,d\theta + \int_{-\pi/2}^{\tan^{-1}(\frac{x+L}{-y})} S_L(\theta)\sin(x+L,y)\,d\theta\ . \quad (3)$$

The coordinates move with the ship, the x-coordinate positive forward and originating at the bow, the y-coordinate positive to starboard and originating at the centerline plane. The amplitude functions $C_o(\theta)$, $S_o(\theta)$, $C_L(\theta)$, and $S_L(\theta)$ are the amplitude of the elementary waves of cosine and sine form originating at the bow and stern respectively. The angle θ is the angle between the direction of motion of the ship and the direction of motion of an elementary wave. The parameter $K_o = g/c^2$, where c is ship speed.

The physical interpretation of Equation (3) is that the elementary waves of any one wave length form a pattern which over part of the surface results from the interference between waves going in directions which are symmetrical on either side of the direction of ship motion. Over the rest of the water surface the pattern results from waves going in only one direction or, for points far enough ahead of the ship and at a fixed distance off the track, there is no pattern at all from the particular wave length. The concept is shown for bow waves in Figure 1. For stern waves a similar pattern originates at the stern.

The terms of Equation (3) may be combined to place it in the form of Equation (1). To do this the following identities are needed:

$$\cos(x+L,y) = \cos(x,y)\cos(K_oL\sec\theta) - \sin(x,y)\sin(K_oL\sec\theta)\ ,$$

$$\sin(x+L,y) = \sin(x,y)\cos(K_oL\sec\theta) + \cos(x,y)\sin(K_oL\sec\theta)\ .$$

With these identities we may make the following definitions:

$$C(\theta) = C_o(\theta) + C_L(\theta)\cos(K_oL\sec\theta) + S_L(\theta)\sin(K_oL\sec\theta),$$
$$-\pi/2 \leqq \theta < \tan^{-1}\left(\frac{x+L}{-y}\right)$$
$$= C_o(\theta), \quad \tan^{-1}\left(\frac{x+L}{-y}\right) \leqq \theta \leqq \tan^{-1}(-x/y);$$

$$S(\theta) = S_o(\theta) - C_L(\theta)\sin(K_oL\sec\theta) + S_L(\theta)\cos(K_oL\sec\theta),$$
$$-\pi/2 \leqq \theta < \tan^{-1}\left(\frac{x+L}{-y}\right),$$
$$= S_o(\theta), \quad \tan^{-1}\left(\frac{x+L}{-y}\right) \leqq \theta \leqq \tan^{-1}(-x/y). \tag{4}$$

The wave height for the free waves originating at the bow and stern may now be placed in the form of Equation (1) by use of these definitions.

It is clear that the terms $C(\theta)$ and $S(\theta)$ derived this way will oscilate even where the original functions $C_o(\theta)$, $S_o(\theta)$, $C_L(\theta)$, and $S_L(\theta)$ are slowly varying functions of θ. On the other hand, if K_oL is of the order of 10 the oscillation will not be particularly rapid. This becomes of importance in examining the errors of the method.

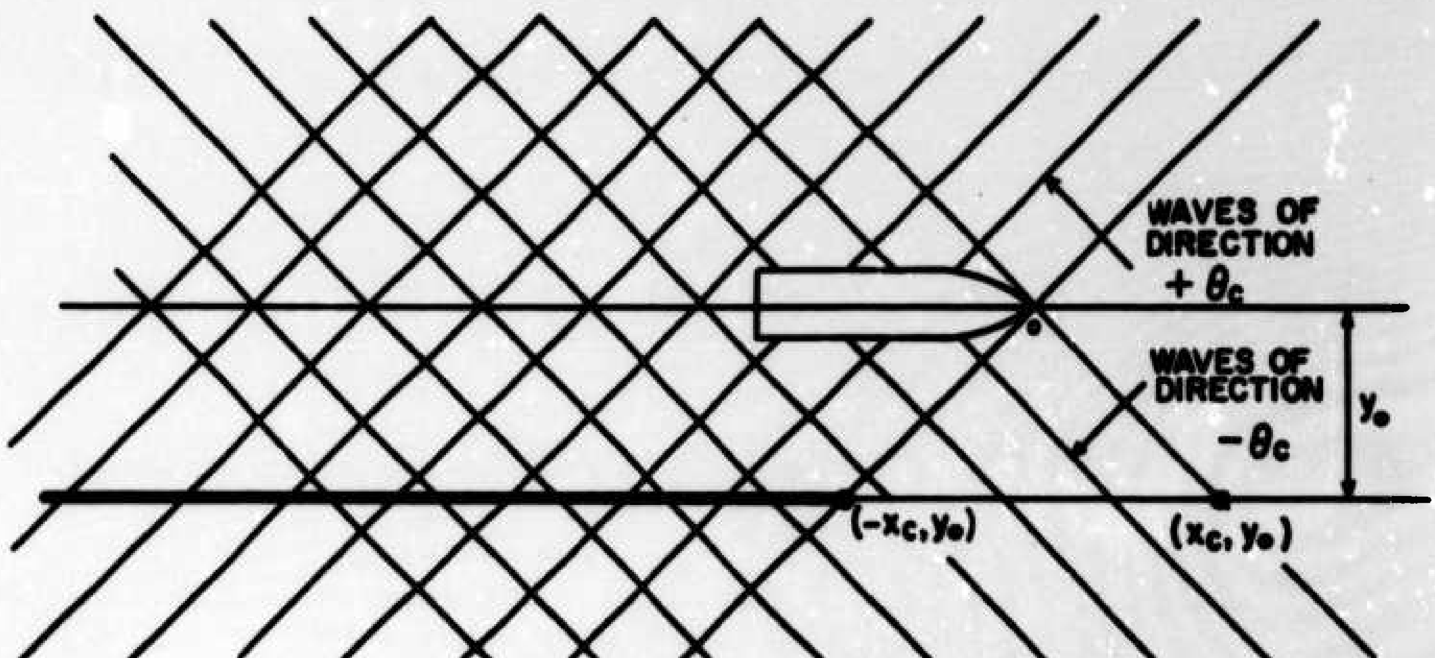

Figure 1

In the next part of the discussion only the case where there is but one point of origin for the waves will be covered. The normal case of the two wave trains -- originating at bow and stern -- will be taken care of by a simple extension of the results for one.

Suppose that we measure the wave height ζ at all points along the line $y = y_0$. We can do this in theory by measuring the wave height at a single point continuously as the ship moves past the point with distance y_0 of closest approach. Then we may decompose the wave height so measured into its elementary waves by taking an exponential transform along the line.

The wave pattern over part of the line $y = y_0$ is different in nature than that over another part. In order to make the results as simple as possible the transform will be taken over a portion of the line which traverses only one kind of pattern. The nature of the difference is clearly shown in Figure 1. Aft of the point $(-x_c, y_0)$ the line $y = y_0$ traverses a wave pattern of waves of length corresponding to the direction θ_0 which is produced by two interfering waves, one traveling in direction θ_c and the other in direction $-\theta_c$. Between the points $(-x_c, y_0)$ and (x_c, y_0) the pattern comes only from waves of direction $-\theta_c$. Forward of (x_c, y_0) there are no waves of direction $-\theta_c$ or θ_c.

Examination of the plot of the function $\theta = \tan^{-1}(-x/y_0)$ reveals a way of dealing separately with the two different kinds of wave pattern. This plot is shown in Figure 2.

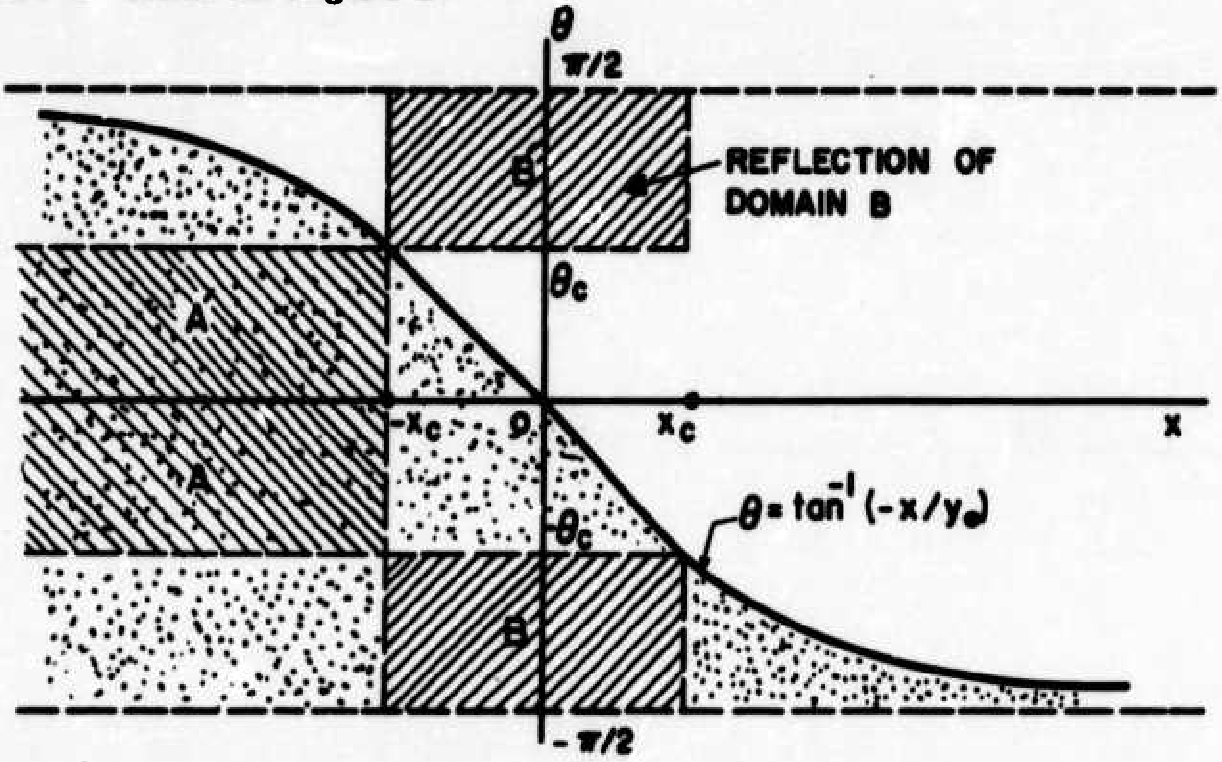

Figure 2.

Suppose we divide the line $y = y_o$ at the point $-x_c = -y_o \tan \theta_c$, where the angle $\theta_c > 0$ is chosen large enough so that most of the wave-making resistance R is given by R_1 in the expression

$$R = R_1 + R_2 , \tag{5}$$

where

$$R_1 = \pi\rho c^2 \int_0^{\theta_c} \left\{[C(\theta)]^2 + [S(\theta)]^2\right\} \cos^3\theta \, d\theta \tag{6}$$

and

$$R_2 = \pi\rho c^2 \int_{\theta_c}^{\pi/2} \left\{[C(\theta)]^2 + [S(\theta)]^2\right\} \cos^3\theta \, d\theta . \tag{7}$$

It is always possible to choose θ_c large enough so that R_2 is as small as we please. On the other hand, if θ_c is made too large the point $-x_c = -y_o \tan \theta_c$ will be so far aft that the wave amplitudes will be small and the accuracy of the measurement will decrease.

Now the wave length of the x-component of a ship wave is known to be $\lambda_\theta = 2\pi/K_o \sec \theta$. Hence if we consider only that portion of the wave height ζ which results from components whose wave length lies between $\lambda_o = 2\pi/K_o$ and $\lambda_{\theta_c} = 2\pi/K_o \sec \theta_c$, we find for all $x < -x_c$ that it is given by

$$\zeta_1 = \int_{-\theta_c}^{\theta_c} C(\theta) \cos(x,y) \, d\theta + \int_{-\theta_c}^{\theta_c} S(\theta) \sin(x,y) \, d\theta . \tag{8}$$

This can be seen by examining Figure 2. The integration is along a vertical line from the lower boundary of region A to the upper boundary of region A' , and the integrand is clearly defined and single-valued in this region. We may use the fact that $C(\theta)$ and $S(\theta)$ are both even functions of θ to simplify the equation further. This gives us

$$\zeta_1 = 2 \int_0^{\theta_c} [C(\theta) \cos(K_o x \sec \theta) + S(\theta) \sin(K_o x \sec \theta)] \cos(K_o y_o \sec \theta \tan \theta) \, d\theta,$$

$$- \infty < x \leqq -x_c . \tag{9}$$

By the use of symmetry we have now restricted the integration to the vertical line from the horizontal axis to the upper boundary of region A' . The domains B and B' are included in Figure 2 to show where a rectangular domain of integration might be laid out which involved those values of θ

which are required to calculate R_2 . Instead of being semi-infinite, that domain is bounded and very small, and it will not be used in the remainder of the derivation.

We must next convert expression (9) into the form of an exponential Fourier transform on a semi-infinite interval. To do this we must first make some changes in variables. Let

$$s(\theta) = K_o \sec \theta , \tag{10}$$

and

$$\beta(\theta) = K_o y_o \sec \theta \tan \theta = s(\theta)\, y_o \tan \theta . \tag{11}$$

Then if we substitute in Equation (9) we get

$$\zeta_1 = 2 \int_0^{\theta_c} [C(\theta) \cos(sx) + S(\theta) \sin(sx)] \cos \beta(\theta)\, d\theta , \quad - \infty < x < -x_c . \tag{12}$$

We now define

$$\begin{aligned} U_1(s) &= C(\theta),\ 0 \leqq \theta \leqq \theta_c,\ K_o \leqq s \leqq K_o \sec \theta_c, \\ &= 0, \quad \theta_c < \theta \leqq \pi/2,\ K_o \sec \theta_c < s < \infty. \\ V_1(s) &= S(\theta),\ 0 \leqq \theta \leqq \theta_c,\ K_o \leqq s \leqq K_o \sec \theta_c \\ &= 0, \quad \theta_c < \theta \leqq \pi/2,\ K_o \sec \theta_c < s < \infty. \end{aligned} \tag{13}$$

Since $d\theta = ds/K_o \sec \theta \tan \theta$, and we may use the notation $\beta[\theta(s)] \equiv \beta$, we can rewrite the expression for ζ_1 as an improper integral in s .

$$\zeta_1 = 2 \int_0^{\infty} \frac{[U_1(s) \cos(sx) + V_1(s) \sin(sx)]}{K_o \sec \theta(s) \tan \theta(s)} \cos \beta\, ds , \quad \text{where } - \infty < x \leqq - x_x . \tag{14}$$

Now we let $x' = x + x_c$. Then it follows that

$$\zeta_1 = 2 \int_0^\infty \{U_1(s)\,[\cos(sx')\cos(sx_c) + \sin(sx')\sin(sx_c)] + V_1(s)\,[\sin(sx')\cos(sx_c) - \cos(sx')\sin(sx_c)]\} \frac{\cos\beta}{K_o \sec\theta(s)\tan\theta(s)}\,ds. \tag{15}$$

Now $U_1(s)$ and $V_1(s)$ are cut off at large s, so we may apply the infinite sine or cosine transform to them. Further, the integrand in Equation (12), from which (13) is derived, is obviously bounded, so the factor $\tan\theta(s)$ in the denominator of Equation (15) cannot produce a singularity. It follows from the properties of the sine and cosine transforms on a semi-infinite interval that we may recover the functions $U_1(s)\cos\beta/K_o\sec\theta(s)\tan\theta(s)$ and $V_1(s)\cos\beta/K_o\sec\theta(s)\tan\theta(s)$ by taking the exponential transform of both sides of Equation (15).

$$\begin{aligned}\int_{-\infty}^{0} \zeta_1(x')\,e^{irx'}\,dx' &= 2 \int_0^\infty\!\!\int_0^\infty \{U_1(s)[\cos(sx')\cos(sx_c) - \sin(sx')\sin(sx_c)] \\ &\qquad + V_1(s)[-\sin(sx')\cos(sx_c) - \cos(sx')\sin(sx_c)]\} \\ &\qquad \times [\cos(rx') - i\sin(rx')]\,\frac{\cos\beta}{K_o\sec\theta(s)\tan\theta(s)}\,ds\,dx' \\ &= \pi\{[U_1(r)\cos(rx_c) - V_1(r)\sin(rx_c)] + i[U_1(r)\sin(rx_c) + V_1(r)\cos(rx_c)]\} \\ &\quad \times \frac{\cos\beta}{K_o\sec\theta(r)\tan\theta(r)}, \quad K_o < r \leqq K_o\sec\theta_c;\ x \leqq -x_c.\end{aligned} \tag{16}$$

We will let $w_1(r) = \int_{-\infty}^{0} \zeta_1(x')\,e^{irx'}\,dx'$. But then the exponential transform of the wave heights over the semi-infinite interval will filter out all components other than those of wave length $2\pi/r$, for $K_o < r \leqq K_o \sec\theta_c$

$$w_1(r) = \int_{-\infty}^{0} \zeta_1(x')\,e^{irx'}dx' = \int_{-\infty}^{0} \zeta(x')\,e^{irx'}dx', \tag{17}$$

where the function $\zeta(x')$ refers to the observed total wave height, not just the components produced by wave lengths more than $2\pi/K_o \sec\theta_c$. We can therefore obtain $w_1(r)$ from the observed wave heights on the line $y = y_o$ at all points aft of $x' = 0$, which is $x = -x_c$.

If we multiply $w_1(r)$ by its complex conjugate $\bar{w}_1(r)$ we get

$$w_1(r)\,\bar{w}_1(r) = \pi^2\{[U_1(r)]^2 + [V_1(r)]^2\}\,\frac{\cos^2\beta}{K_o^2\sec^2\theta(r)\tan^2\theta(r)},$$
$$\text{where } K_o < r \leqq K_o\sec\theta_c. \tag{18}$$

This result is clearly independent of the origin chosen for the coordinate x' provided all points aft of it have a wave pattern resulting from the superposition of symmetrical waves going in directions θ and $-\theta$ for all values of $r(|\theta|)$ under consideration.

We may now return to the variable θ and employ Equation (6) to get the resistance corresponding to a single train of waves.

$$R_1 = K_o^2 \rho \frac{c^2}{\pi} \int_0^{\theta_c} w_1(K_o \sec\theta)\, \bar{w}_1(K_o \sec\theta) \frac{\sin\theta \tan\theta}{\cos^2\beta(\theta)} d\theta . \quad (19)$$

Equation (19) provides an approximation to the resistance which is correct, except for the omission of the resistance component R_2, for a hull which produces a single wave train. It may be extended to the case where the ship produces two or more wave trains by the simple expedient of making the upper limit of integration small enough so that only such portions of both bow and stern waves are included in the calculation as are formed by interference between waves moving in both $+\theta$ and $-\theta$ directions. This can be done because Equation (18) is independent of the origin of x'. To accomplish the extension to the general case all that need be done is to make the upper limit of the integral of Equation (19) the quantity $\tan^{-1}(\frac{x_c-L}{y_o})$ instead of θ_c. The result is

$$R_1 = \frac{K_o^2 \rho\, c^2}{\pi} \int_0^{\tan^{-1}(\frac{x_c-L}{y_o})} \frac{w_1(K_o \sec\theta)\, \bar{w}_1(K_o \sec\theta) \sin\theta \tan\theta \, d\theta}{\cos^2\beta(\theta)} \quad (20)$$

<u>Proof of the Corollary:</u>

Equation (20) provides an approximation R_1 for the wave-making resistance R, and the error of the approximation is given by Equation (7) as the qunatity R_2 provided the lower limit of the integration for R_2 is made the same as the upper limit of the integral for R_1 in Equation (20). However, it can be shown that the quantity $\{[C(\theta)]^2 + [S(\theta)]^2\} \cos^3\theta$ in Equation (7) goes to zero as θ approaches $\pi/2$ like $\sec^n\theta\, e^{-k \sec^2\theta}$, where k is a positive real constant and n is a real constant. Since this is true, it follows that there exists some $\varepsilon > 0$ such that $R_2 < c$ for all $\frac{\pi}{2} - \varepsilon \leqq \theta_c \leqq \frac{\pi}{2}$, where ε is any number greater than zero. But from this it follows that the approximation R_1 may be made arbitratily close to R by choosing $\frac{x_c-L}{y_o}$ large enough, and so the corollary is proved.

III. Examination of the Errors and Possible Utility of the Method

In part II it was shown that it is possible in theory to find the wave-making resistance of a ship by measuring the wave height at all points aft of a ship on a line parallel to its track. It remains to be shown, however, that this method can be applied. In order to get a feel for the problems and possibilities, it is necessary to explore the sources of error in the method. This exploration, together with conclusions on how best to apply the method, will be discussed in this part of the paper.

1. Error from Disregarding R_2

It is possible to conclude by examining curves of $C(\theta)$ and $S(\theta)$ produced by other authors,[2,3] that nearly all the resistance is included in R_1 with less than one percent in R_2 when the upper limit of the integral is chosen as 80° of arc. Since this can always be arranged by making x_c large enough, it is clear that this source of error can be controlled. To permit the upper limit of the integral to be 80° it is necessary that the point $(-x_c + L, y_o)$, which is the forward end of the line segment on which the observations are to be made, subtend an angle of no more than 10 degrees with the track of the ship when viewed from the ship's stern.

2. Error from the Factor $\cos^2\beta$ in the Denominator of Equation (20)

The factor $\cos^2\beta = \cos^2[K_o y_o \sec\theta \tan\theta]$ in the denominator of Equation (20), the expression for the resistance, will have zeros in the range $0 \leq \theta \leq \pi/2$. Although there should be corresponding zeros in the numerator, the values in the numerator will be obtained from experiment and they certainly cannot be trusted to come out to be exactly zero at the points where they theoretically should. Their existence is a consequency of the interference by the elementary waves going in the direction θ with those going in the direction $-\theta$. The nature of the interference pattern is shown in Figure 3.

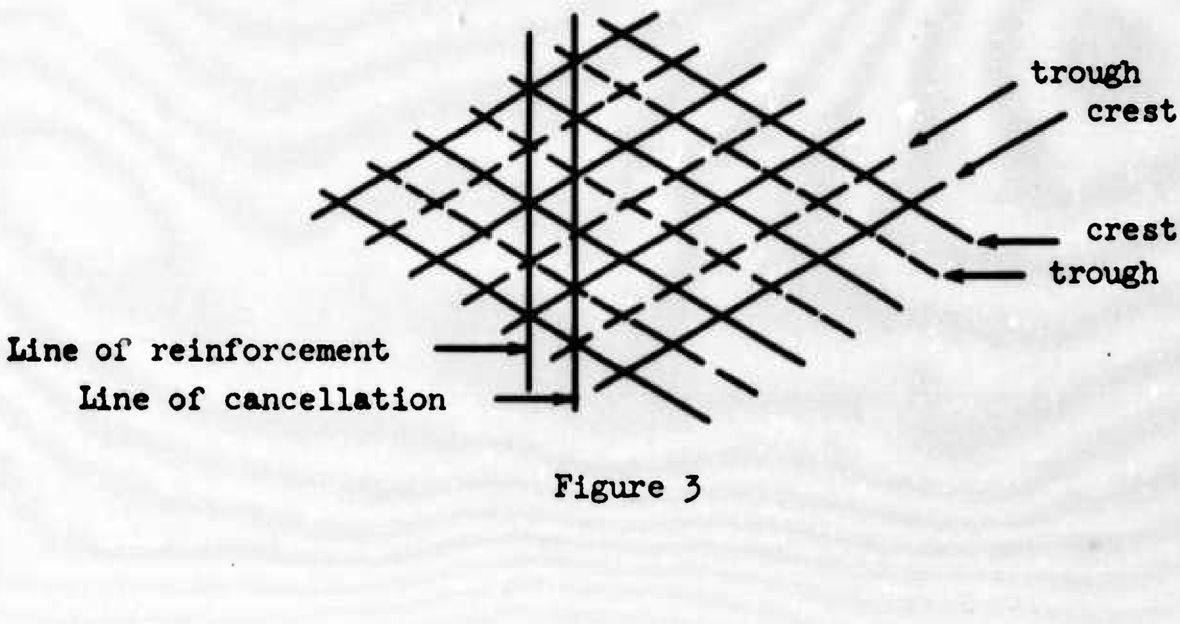

Figure 3

To get around this problem, we may take advantage of the continuity of $C(\theta)$ and $S(\theta)$ to continue the integrand across the zeros. If we know enough of the shape of the integrand of Equation (20) to be sure that it is a slowly varying function of θ in the vicinity of the zeros of $\cos\beta$, then we need only evaluate it for points where $\cos\beta$ is not close to zero and fair the curve across the points where $\cos\beta = 0$. On the other hand, if we have two lines parallel to the ship's track and they do not have the same absolute value of y_o then it is easy to show that nearly all the zeros of $\cos\beta$ on one line will not be zeros on the other line. A set of measurements on a second line parallel to the ship's track is therefore an alternate solution to this difficulty.

It might seem reasonable to solve the problem by making y_o so small that the first zero of $\cos\beta$ would occur for $\theta > 80°$. Unfortunately, this makes y_o so small that the line on which the measurements are to be made will nearly always fall in the ship's wake. Since the derivation in this paper does not take into account the effect on the wave pattern in the wake of the motion of the wake, this solution does not seem to be an acceptable one. Either of the other two, however, should be satisfactory.

3. Error from Use of a Finite Rather than a Semi-Infinite Line Segment on Which to Measure Wave Heights

The derivation of part II of this paper assumes that the ship has been in motion an infinitely long time and that there is an infinitely long train of waves behind the ship on which the measurement of wave height can be made. This cannot even be approximated in a model basin. Hence we must examine the error to be expected from making the measurement on a finite segment of the line aft of $x = -x_c$ rather than on a semi-infinite segment.

To find the order of magnitude of the errors to be expected we start with Equation (16), but instead of using the lower limit $-\infty$ for the integration with respect to x' we use the lower limit $-X'$, where $-X'$ is the distance from the point $x = -x_c$ to the after end of the line segment on which the wave height is measured. After some trigonometric substitutions this gives us the following equation:

$$\int_{-X'}^{0} \zeta_1(x')\, e^{irx'}\, dx' =$$

$$= \int_{s=K_o}^{\infty} \left\{U_1(s)\cos(sx_c) - V_1(s)\sin(sx_c) + iU_1(s)\sin(sx_c) + iV_1(s)\cos(sx_c)\right\}$$

$$\times \frac{\cos\beta}{\sec\theta(s)\tan\theta(s)} \cdot \frac{\sin[(s-r)X']}{(s-r)X'}\, d[(s-r)X']$$

$$+ 2 \int_{s=K_o}^{\infty} \{-U_1(s) \sin(sx_c) - V_1(s) \cos(sx_c) + iU_1(s) \cos(sx_c) - iV_1(s) \sin(sx_c)\}$$

$$\times \frac{\cos \beta}{\sec \Theta(s) \tan \Theta(s)} \cdot \frac{\sin^2[\frac{1}{2}(s-r)X']}{\frac{1}{2}(s-r)X'} \, d[\tfrac{1}{2}(s-r)X']$$

$$+ 2 \int_{s=K_o}^{\infty} \{-U_1(s) \sin(sx_c) - V_1(s) \cos(sx_c) - iU_1(s) \cos(sx_c) + iV_1(s) \sin(sx_c)\}$$

$$\times \frac{\cos \beta}{\sec \Theta(s) \tan \Theta(s)} \cdot \frac{\sin^2[\frac{1}{2}(s+r)X']}{\frac{1}{2}(s+r)X'} \, d[\tfrac{1}{2}(s+r)X']$$

$$+ \int_{s=K_o}^{\infty} \{U_1(s) \cos(sx_c) - V_1(s) \sin(sx_c) - iU_1(s) \sin(sx_c) - iV_1(s) \cos(sx_c)\}$$

$$\frac{\cos \beta}{\sec \Theta(s) \tan \Theta(s)} \cdot \frac{\sin[(s+r)X']}{(s+r)X'} \, d[(s+r)X'] \; . \tag{21}$$

The first of the four integrals on the right side of this equation tends as X' becomes infinite to the inverse transform given by Equation (16). The sum of the remaining three integrals must therefore go to zero. That this is true not only for their sum but for each of them separately is clear. The second integral is like the first one but multiplied by a factor $\sin[\frac{1}{2}(s-r)X']$. This makes the integrand an odd function of $(s-r)X'$ which goes to zero as $a \to r$ and which for large X' goes to zero for all s not close to r . Hence for large X' it goes to zero. The absolute value of the third and fourth integrals is dominated by $2\{\text{Max}|U_1|+|V_1|\}$ $[\frac{\pi}{2} - \text{Si}(K_oX')]$, which also goes to zero as X' becomes infinite.

To estimate the error which corresponds to a given value of X' we may, of course, go through the integrations of Equation (21) if we know the approximate shape of the functions $U_1(s)$ and $V_1(s)$. We may then compare the result with the corresponding result with the same functions using Equation (16), and then see what effect the difference would have on the resistance as calculated by Equation (20). We can also get some idea of the order of magnitude of the errors to be expected by approximations, and better still, we may set some rules for the proper relative magnitudes of quantities in the measurement.

The first integral of Equation (21) is an approximation to the inverse transform only when X' is very large. We may see this by examining the nature of the approximation for the first term of the first integral.

The approximation assumes

$$\int_{s=K_o}^{\infty} \frac{U_1(s)\cos(sx_c)\cos\beta}{\sec\Theta(s)\tan\Theta(s)} \quad \frac{\sin[(s-r)X']}{(s-r)X'} \quad d[(s-r)X']$$

$$\approx \frac{U_1(r)\cos(rx_c)\cos\{\beta[\Theta(r)]\}}{\sec\Theta(r)\tan\Theta(r)} \int_{-\infty}^{\infty} \frac{\sin[(s-r)X']}{(s-r)X'} \quad d[(s-r)X']$$

$$= \frac{\pi\, U_1(r)\cos(rx_c)\cos\{\beta[\Theta(r)]\}}{\sec\Theta(r)\tan\Theta(r)} . \tag{22}$$

For this approximation to be reasonably accurate it is necessary that the factor $\sin[(s-r)X']$ oscillate very rapidly compared with the rate of change of the rest of the integrand. We may therefore profitably investigate the period of the other oscillatory factors in the integrand.

For $\Theta < 80°$, $\tan\Theta < 6$. Hence $\cos\beta$ oscillates no faster than $\cos(6y_os)$ over the range in which we are interested. Hence we may reasonably require that $X'K_o \gg 6y_oK_o$, and hence $X' \gg 6y_o$. Further, if the amplitude functions C_o, S_o, C_L and S_L are slowly varying functions of Θ then the oscillation of the remaining terms of the integrand may be controlled by the factor $\cos(sx_c)$ or $\sin(sx_c)$. But if this is true we must also have $X' \gg x_c$. Since $x_c > L$, this means $X' \gg L$.

Compared with the possible error in the first integral of Equation (21) it appears probable that contributions from the second, third, and fourth integrals will be quite small for fairly reasonable values of K_oX' . For example, the third and fourth integrals would probably contribute less than one percent of the first for $K_oX' = 100$, which is only ten ship lengths for an 18 knot ship 300 feet long. The second integral goes to zero as $1/(K_oX')^2$, and will also disappear rapidly. The first one, however, can have errors of the order of several percent from the appraximation which is supposed to be equal to it even for runs quire a bit longer than ten ship lengths. It is unfortunate that there seems no way which does not involve considerable labor to place a bound on this error.

4. Some Tentative Rules for Application of the Method

It seems reasonable from the discussion so far to specify that the line on which the measurement is made should be as close to the track of the ship as possible without getting into the wake. The foreward end of the

line should subtend an angle of about 10 degrees of arc with the track of the ship when viewed from its stern. The line should extend as far aft as possible in order to minimize error from use of a finite rather than a semi-infinite line. Finally, it is obvious that reflections from the walls of a towing tank would introduce serious complications and would certainly invalidate the formulae derived in this paper. Hence for these formulae, at least, the measurement must be made in a very wide and very long tank.

IV. Conclusions

The method of measuring wave-making resistance by observing wave height at all points on a line parallel to the track of a ship from a point a short distance aft of the ship to a point infinitely far aft is theoretically accurate. This method is equivalent to measuring the wave height at one point as the ship passes it on a straight course. If the line on which the wave height is known is of only a finite length then the accuracy of the result decreases. Although a bound on the error can be calculated if the shape of the curves $C(\theta)$ and $S(\theta)$ is known approximately, it is not a simple calculation. Unfortunately, the accuracy will probably be enough affected by using runs of a length which can be obtained in a maneuvering basin so that the calculation of the error is a necessary one. This matter aside, the technique looks simple and attractive. Certainly this method or an equivalent appear to be that best suited for the measurement of the wave-making resistance of a full-scale ship in open water or of a self-propelled model on a pond, so it seems to merit further investigation.

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THE MOTION OF A SHIP IN RESTRICTED WATER

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INTRODUCTION

In the paper we develop a theory of the motion of a ship under the action of a constant propeller thrust and rudder force in restricted but calm water. We assume that the ship moves through a canal with variable breadth and depth. We shall give a perturbation theory with three perturbation parameters ϵ_1, ϵ_2 and ϵ_3, analogously to the famous work of Peters and Stoker[9] on the motion of a ship in a seaway. ϵ_1 is the beam-length ratio of the ship, ϵ_2 is the maximum elevation of the bottom above a horizontal bottom and ϵ_3 is the maximum deviation of the canal walls from parallel vertical walls. Under constant propeller thrust and suitable rudder force the main motion of the ship is a translation with constant velocity s. Generally our problem is non-stationary because of the variable breadth and depth of the canal. Only in cases where the deviation of the bottom and tank walls from the horizontal bottom and vertical parallel walls is independent of the coordinate in whose direction the main motion of the ship takes place, the problem becomes stationary referring to a coordinate system which translates with constant velocity s. It is clear that in the last cases the cross section of the canal is independent of the same coordinate. Therefore our theory includes also the cases where the ship moves with constant velocity in the direction of the centerline of a canal of trapezoidal cross section. More information will be given later.

1) Formulation of the Problem

Let x, y, z be a rectangular coordinate system, which is fixed in space (Figure 1). We assume the main motion of the ship to be a translation with constant velocity S_{000} in the direction of the x-axis. The xy-plane of our coordinate system may always coincide with the undisturbed free surface of water and the z-axis will be directed vertically upwards.

Further we assume an inviscid, incompressible and irrotational flow, so that there will exist a velocity potential $\phi(x,y,z,t)$. For later linearization we consider a three parameters family of motions with parameters ϵ_1, ϵ_2, ϵ_3 where ϵ_1 is the beam-length ratio of the ship, ϵ_2 the maximum elevation of the bottom above the horizontal bottom and ϵ_3 the maximum deviation of the tank walls from vertical and parallel walls.

In our coordinate system the equation of the free surface then takes the form:

$$z = \zeta(x,y,t;\epsilon_1,\epsilon_2,\epsilon_3) \quad . \tag{1.01}$$

Under our restrictions we have for the velocity field

$$\mathfrak{w}(x,y,z,t;\epsilon_1,\epsilon_2,\epsilon_3) = (v_x(x,y,z,t;\epsilon_1,\epsilon_2,\epsilon_3), v_y(x,y,z,t;\epsilon_1,\epsilon_2,\epsilon_3), v_z(x,y,z,t;\epsilon_1,\epsilon_2,\epsilon_3))$$

$$= \nabla\phi(x,y,z,t;\epsilon_1,\epsilon_2,\epsilon_3) \tag{1.02}$$

where

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \tag{1.03}$$

and the function ϕ satisfies Laplace's equation

$$\nabla^2\phi = \phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \tag{1.04}$$

in the domain occupied by the fluid. If respectively $p(x,y,z,t;\epsilon_1,\epsilon_2,\epsilon_3)$, g and ϱ are the fluid pressure, gravity acceleration and fluid density, in the domain occupied by the fluid holds the equation of Bernoulli:

$$\frac{p}{\varrho} + gz + \phi_t + \frac{1}{2}(\nabla\phi)(\nabla\phi) = 0 \tag{1.05}$$

With this equation the pressure $p(x,y,z,t;\ \epsilon_1,\epsilon_2,\epsilon_3)$ can be determined if the potential function $\phi(x,y,z,t;\ \epsilon_1,\epsilon_2,\epsilon_3)$ is known. If we introduce the hydrodynamical pressure $P(x,y,z,t;\ \epsilon_1,\epsilon_2,\epsilon_3)$ defined by

$$P(x,y,z,t;\varepsilon_1,\varepsilon_2,\varepsilon_3) = \frac{1}{g}\,p(x,y,z,t;\varepsilon_1,\varepsilon_2,\varepsilon_3) + gz \qquad (1.06)$$

the Bernoulli equation (1.5) becomes

$$P + \phi_t + \frac{1}{2}(\nabla\phi)(\nabla\phi) = 0 \qquad (1.07)$$

Now we are able to derive the boundary conditions on the free surface. At the free surface we have the dynamical condition

$$p(x,y,z,t;\varepsilon_1,\varepsilon_2,\varepsilon_3) = \text{const.} \quad \text{on} \quad z = \zeta(x,y,t;\varepsilon_1,\varepsilon_2,\varepsilon_3)$$

which results on account of (1.5) in

$$g\cdot\zeta(x,y,t;\varepsilon_1,\varepsilon_2,\varepsilon_3) + \phi_t(x,y,z,t;\varepsilon_1,\varepsilon_2,\varepsilon_3) + (\nabla\phi)\cdot(\nabla\phi) = 0 \qquad (1.08)$$

$$\text{on} \quad z = \zeta(x,y,t;\varepsilon_1,\varepsilon_2,\varepsilon_3).$$

If now

$$F(x,y,z,t;\varepsilon_1,\varepsilon_2,\varepsilon_3) = 0 \qquad (1.09)$$

is the equation of a rigid surface, which is a boundary of our fluid domain, on this surface we have the kinematic condition

$$\frac{dF}{dt} = \phi_x F_x + \phi_y F_y + \phi_z F_z + F_t = 0 \quad \text{on} \quad F(x,y,z,t;\varepsilon_1,\varepsilon_2,\varepsilon_3) = 0 \qquad (1.10)$$

If specifically

$$F(x,y,z,t;\varepsilon_1,\varepsilon_2,\varepsilon_3) \equiv z - \zeta(x,y,t;\varepsilon_1,\varepsilon_2,\varepsilon_3) \qquad (1.11)$$

we get as kinematic condition on the free surface

$$-\phi_x\cdot\zeta_x - \phi_y\zeta_y + \phi_z - \zeta_t = 0 \quad \text{on} \quad z = \zeta(x,y,t;\varepsilon_1,\varepsilon_2,\varepsilon_3) \qquad (1.12)$$

In order to linearize the above nonlinear boundary conditions we assume that all the functions which appear in connection with our problem can be expanded in asymptotically convergent power series in terms of the three perturbation parameters ϵ_1, ϵ_2, ϵ_3. With respect to the physical meaning of ϵ_1, ϵ_2, ϵ_3 we can assume the following expansions:

$$\phi(x,y,z,t;\epsilon_1,\epsilon_2,\epsilon_3) = \epsilon_1\phi_{100}(x,y,z,t) + \epsilon_1\epsilon_2\phi_{110}(x,y,z,t) + \epsilon_1\epsilon_3\phi_{101}(x,y,z,t) + \epsilon_1\epsilon_2\epsilon_3\phi_{111}(x,y,z,t) + \ldots \tag{1.13}$$

$$\zeta(x,y,t;\epsilon_1,\epsilon_2,\epsilon_3) = \zeta_{000}(x,y,t) + \epsilon_1\zeta_{100}(x,y,t) + \epsilon_1\epsilon_2\zeta_{110}(x,y,t) + \epsilon_1\epsilon_3\zeta_{101}(x,y,t) + \epsilon_1\epsilon_2\epsilon_3\zeta_{111}(x,y,t) + \cdots \tag{1.14}$$

$$\begin{aligned} F(x,y,z,t;\epsilon_1,\epsilon_2,\epsilon_3) = {} & F_{000}(x,y,z,t) + \epsilon_1 F_{100}(x,y,z,t) + \epsilon_2 F_{010}(x,y,z,t) \\ & + \epsilon_3 F_{001}(x,y,z,t) + \epsilon_1\epsilon_2 F_{110}(x,y,z,t) \\ & + \epsilon_1\epsilon_3 F_{101}(x,y,z,t) + \epsilon_2\epsilon_3 F_{011}(x,y,z,t) \\ & + \epsilon_1\epsilon_2\epsilon_3 F_{111}(x,y,z,t) + \cdots \end{aligned} \tag{1.15}$$

$$\begin{aligned} p(x,y,z,t;\epsilon_1,\epsilon_2,\epsilon_3) = {} & p_{000}(x,y,z,t) + \epsilon_1 p_{100}(x,y,z,t) \\ & + \epsilon_1\epsilon_2 p_{110}(x,y,z,t) + \epsilon_1\epsilon_3 p_{101}(x,y,z,t) \\ & + \epsilon_1\epsilon_2\epsilon_3 p_{111}(x,y,z,t) + \cdots \end{aligned} \tag{1.16}$$

where for the last expansion we have used (1.7) and (1.13). If we now substitute in equation (1.7) the expansions (1.13) and (1.16) we get collecting the coefficients of the first powers of ϵ_1, ϵ_2, ϵ_3 successively:

$$p_{000}(x,y,z,t) = 0 \tag{1.17}$$

$$p_{100}(x,y,z,t) + \phi_{100\,t}(x,y,z,t) = 0 \tag{1.18}$$

$$p_{110}(x,y,z,t) + \phi_{110\,t}(x,y,z,t) = 0 \tag{1.19}$$

$$p_{101}(x,y,z,t) + \phi_{101\,t}(x,y,z,t) = 0 \tag{1.20}$$

$$p_{111}(x,y,z,t) + \phi_{111\,t}(x,y,z,t) = 0 \tag{1.21}$$

Further the substitution of the power series (1.13) into the potential equation (1.4) gives:

$$\nabla^2\phi_{100} = 0\,,\quad \nabla^2\phi_{110} = 0\,,\quad \nabla^2\phi_{101} = 0\,,\quad \nabla^2\phi_{111} = 0,\ \ldots \tag{1.22}$$

Here the question necessarily arises as to what are the domains in which the equations (1.17) to (1.22) are valid. Therefore, we have first to consider the boundary conditions on the free surface. If we substitute the power series (1.13) and (1.14) in the boundary conditions (1.8) and (1.10) we get for the coefficients of the corresponding powers of ϵ_1, ϵ_2, ϵ_3:

$$\begin{aligned} g\cdot\zeta_{000}(x,y,t) &= 0\\ \zeta_{000\,t}(x,y,t) &= 0 \end{aligned} \tag{1.23}$$

$$\begin{aligned} g\cdot\zeta_{100}(x,y,t) + \phi_{100\,t}(x,y,0,t) &= 0\\ \phi_{100\,z}(x,y,0,t) - \zeta_{100\,t}(x,y,t) &= 0 \end{aligned} \tag{1.24}$$

$$\begin{aligned} g\cdot\zeta_{110}(x,y,t) + \phi_{110\,t}(x,y,0,t) &= 0\\ \phi_{110\,z}(x,y,0,t) - \zeta_{110\,t}(x,y,t) &= 0 \end{aligned} \tag{1.25}$$

$$\begin{aligned} g\cdot\zeta_{101}(x,y,t) + \phi_{101\,t}(x,y,0,t) &= 0\\ \phi_{101\,z}(x,y,0,t) - \zeta_{101\,t}(x,y,t) &= 0 \end{aligned} \tag{1.26}$$

$$\begin{aligned} g\cdot\zeta_{111}(x,y,t) + \phi_{111\,t}(x,y,0,t) &= 0\\ \phi_{111\,z}(x,y,0,t) - \zeta_{111\,t}(x,y,t) &= 0 \end{aligned} \tag{1.27}$$

Eliminating the functions $\zeta_{ijk}(x,y,t)$ gives the following boundary conditions on the free surface:

$$g\,\phi_{100z}(x,y,0,t) + \phi_{100tt}(x,y,0,t) = 0 \qquad (1.28)$$

$$g\,\phi_{110z}(x,y,0,t) + \phi_{110tt}(x,y,0,t) = 0 \qquad (1.29)$$

$$g\,\phi_{101z}(x,y,0,t) + \phi_{101tt}(x,y,0,t) = 0 \qquad (1.30)$$

$$g\,\phi_{111z}(x,y,0,t) + \phi_{111tt}(x,y,0,t) = 0 \qquad (1.31)$$

Now we consider the boundary conditions on the bottom of the tank. Let

$$z + h - \varepsilon_2\, g(x,y) = 0 \qquad (1.32)$$

be the equation of the bottom, then the kinematic condition (1.10) gives

$$-\varepsilon_2\cdot\phi_x\cdot g_x - \varepsilon_2\cdot\phi_y\cdot g_y + \phi_z = 0 \quad \text{on} \quad z = -h + \varepsilon_2\cdot g(x,y) \qquad (1.33)$$

Substitution of the power series (1.13) and equating the coefficients of corresponding powers of ϵ_1, ϵ_2, ϵ_3 to zero gives:

$$\phi_{100z}(x,y,-h,t) = 0 \qquad (1.34)$$

$$\begin{aligned}\phi_{110z}(x,y,-h,t) = {} & \phi_{100x}(x,y,-h,t)\cdot g_x(x,y) \\ & +\phi_{100y}(x,y,-h,t)\cdot g_y(x,y) \\ & -\phi_{100zz}(x,y,-h,t)\cdot g(x,y)\end{aligned} \qquad (1.35)$$

$$\phi_{101z}(x,y,-h,t) = 0 \qquad (1.36)$$

$$\begin{aligned}\phi_{111z}(x,y,-h,t) = {} & \phi_{101x}(x,y,-h,t)\, g_x(x,y) \\ & +\phi_{101y}(x,y,-h,t)\, g_y(x,y) \\ & -\phi_{101zz}(x,y,-h,t)\, g(x,y)\end{aligned} \qquad (1.37)$$

Last we consider the boundary conditions on the canal walls. Let

$$y - \mathfrak{B}_1 + \epsilon_3 \cdot b_1(x,z) = 0 \quad \text{and} \quad y + \mathfrak{B}_2 - \epsilon_3 b_2(x,z) = 0 \tag{1.38}$$

be the equations of the canal walls. The kinematic condition (1.10) then gives:

$$\pm \epsilon_3 \phi_x \cdot b_{ix} + \phi_y \pm \epsilon_3 \phi_z \cdot b_{iz} = 0 \quad \text{on} \quad y = \pm \mathfrak{B}_i \mp \epsilon_3 b_i(x,z), \quad i = 1,2 \tag{1.39}$$

where we have the upper sign for i = 1 and the lower sign for i = 2. Substitution of the power series (1.13) and equating the coefficients of corresponding powers of ϵ_1, ϵ_2, ϵ_3 to zero gives for i = 1, 2:

$$\phi_{100y}(x, \pm\mathfrak{B}_i, z, t) = 0 \tag{1.40}$$

$$\phi_{110y}(x, \pm\mathfrak{B}_i, z, t) = 0 \tag{1.41}$$

$$\begin{aligned} \phi_{101y}(x, \pm\mathfrak{B}_i, z, t) = & \mp \phi_{100x}(x, \pm\mathfrak{B}_i, z, t) \cdot b_{ix}(x,z) \\ & \mp \phi_{100z}(x, \pm\mathfrak{B}_i, z, t) \cdot b_{iz}(x,z) \\ & \pm \phi_{100yy}(x, \pm\mathfrak{B}_i, z, t) \cdot b_i(x,z) \end{aligned} \tag{1.42}$$

$$\begin{aligned} \phi_{111y}(x, \pm\mathfrak{B}_i, z, t) = & \mp \phi_{110x}(x, \pm\mathfrak{B}_i, z, t) \cdot b_{ix}(x,z) \\ & \mp \phi_{110z}(x, \pm\mathfrak{B}_i, z, t) \cdot b_{iz}(x,z) \\ & \pm \phi_{110yy}(x, \pm\mathfrak{B}_i, z, t)\, b_i(x,z) \end{aligned} \tag{1.43}$$

Additionally there are boundary conditions on the ship's surface and suitable radiation conditions which will be discussed later. For our problem it is more appropriate to formulate boundary value problems

for the hydrodynamic pressures $p_{ijk}(x,y,z,t)$ instead of the potentials $\phi_{ijk}(x,y,z,t)$. Equations (1.18) to (1.21) now give

$$p_{1jk}(x,y,z,t) = -\phi_{1jk\,t}(x,y,z,t) \qquad j = 0,1 \tag{1.45}$$

and from (1.22) follows: $\qquad k = 0,1$

$$\nabla^2 p_{1jk} = p_{1jk\,xx} + p_{1jk\,yy} + p_{1jk\,zz} = 0 \qquad \begin{matrix} j = 0,1 \\ k = 0,1 \end{matrix} \tag{1.46}$$

Therefore we get successively the following boundary value problems:

$$\nabla^2 p_{100} = p_{100\,xx} + p_{100\,yy} + p_{100\,zz} = 0 \qquad \begin{cases} -h < z < 0 \\ -B_2 < y < B_1 \\ \text{outside the ship} \end{cases}$$

$$p_{100\,z}(x,y,-h,t) = 0$$

$$p_{100\,y}(x,\pm B_i,z,t) = 0 \qquad i = 1,2$$

$$g\,p_{100\,z}(x,y,0,t) + p_{100\,tt}(x,y,0,t) = 0 \quad ;$$

$$\nabla^2 p_{110} = p_{110\,xx} + p_{110\,yy} + p_{110\,zz} = 0 \qquad \begin{cases} -h < z < 0 \\ -B_2 < y < B_1 \\ \text{outside the ship} \end{cases}$$

$$\begin{aligned} p_{110\,z}(x,y,-h,t) = \; & p_{100\,x}(x,y,-h,t)\cdot g_x(x,y) \\ & + p_{100\,y}(x,y,-h,t)\cdot g_y(x,y) \\ & - p_{100\,zz}(x,y,-h,t)\cdot g(x,y) \end{aligned}$$

$$p_{110\,y}(x,\pm B_i,z,t) = 0 \qquad i = 1,2$$

$$g\,p_{110\,z}(x,y,0,t) + p_{110\,tt}(x,y,0,t) = 0$$

$$\nabla^2 \varphi_{101} = \varphi_{101xx} + \varphi_{101yy} + \varphi_{101zz} = 0 \qquad \begin{cases} -h < z < 0 \\ -B_2 < y < B_1 \\ \text{outside the ship} \end{cases}$$

$$\varphi_{101z}(x,y,-h,t) = 0$$

$$\begin{aligned} \varphi_{101y}(x,\pm B_i,z,t) = &\mp \varphi_{100x}(x,\pm B_i,z,t)\cdot b_{ix}(x,z) \\ &\mp \varphi_{100z}(x,\pm B_i,z,t)\cdot b_{iz}(x,z) \\ &\pm \varphi_{100yy}(x,\pm B_i,z,t)\cdot b_i(x,z) \qquad i=1,2 \end{aligned}$$

$$g\,\varphi_{101z}(x,y,0,t) + \varphi_{101tt}(x,y,0,t) = 0$$

$$\nabla^2 \varphi_{111} = \varphi_{111xx} + \varphi_{111yy} + \varphi_{111zz} = 0 \qquad \begin{cases} -h < z < 0 \\ -B_2 < y < B_1 \\ \text{outside the ship} \end{cases}$$

$$\begin{aligned} \varphi_{111z}(x,y,-h,t) = &\ \varphi_{101x}(x,y,-h,t)\cdot g_x(x,y) \\ &+ \varphi_{101y}(x,y,-h,t)\cdot g_y(x,y) \\ &- \varphi_{101zz}(x,y,-h,t)\cdot g(x,y) \end{aligned}$$

$$\begin{aligned} \varphi_{111y}(x,\pm B_i,z,t) = &\mp \varphi_{110x}(x,\pm B_i,z,t)\cdot b_{ix}(x,z) \\ &\mp \varphi_{110z}(x,\pm B_i,z,t)\cdot b_{iz}(x,z) \\ &\pm \varphi_{110yy}(x,\pm B_i,z,t)\cdot b_i(x,z) \qquad i=1,2 \end{aligned}$$

$$g\,\varphi_{111z}(x,y,0,t) + \varphi_{111tt}(x,y,0,t) = 0$$

Additionally there are boundary conditions on the ship's surface and suitable radiation conditions. For the functions $\zeta_{1jk}(x,y,t)$ we obtain from (1.24) - (1.27):

$$\zeta_{1jk}(x,y,t) = -\frac{1}{g}\,\varphi_{1jk}(x,y,0,t) \qquad \begin{matrix} j=0,1 \\ k=0,1 \end{matrix} .$$

2. Discussion of the Several Coordinate Systems Used

Since we deal with a moving rigid body it is convenient, to refer the motion to the various types of moving coordinate systems as well as to the fixed x,y,z-coordinate system. Let $\bar{x}$, $\bar{y}$, $\bar{z}$ be the first moving coordinate system with the following properties: The $\bar{x},\bar{y}$-plane coincides with the x,y-plane of the fixed coordinate system, the $\bar{z}$-axis is vertically upward and contains the center of gravity of the ship. The $\bar{x}$-axis has always the direction of the horizontal component of the velocity of the center of gravity of the ship (Figure 1). If now

$$\mathfrak{R} = \left(x_c(t;\epsilon_1,\epsilon_2,\epsilon_3),\, y_c(t;\epsilon_1,\epsilon_2,\epsilon_3),\, z_c(t;\epsilon_1,\epsilon_2,\epsilon_3)\right)$$

is the position vector of the center of gravity of the ship relative to the fixed coordinate system,

$$\dot{\mathfrak{R}} = \left(\dot{x}_c(t;\epsilon_1,\epsilon_2,\epsilon_3),\, \dot{y}_c(t;\epsilon_1,\epsilon_2,\epsilon_3),\, \dot{z}_c(t;\epsilon_1,\epsilon_2,\epsilon_3)\right)$$

is the velocity of the center of gravity and the $\bar{x}$-axis has the direction of the velocity vector

$$\vec{u} = \dot{x}_c(t;\epsilon_1,\epsilon_2,\epsilon_3)\mu_x + \dot{y}_c(t;\epsilon_1,\epsilon_2,\epsilon_3)\mu_y \tag{2.01}$$

with μ_x, μ_y unit vectors in the direction of the x- and y-axis. If $\mu_{\bar{x}}$ is the unit vector along the $\bar{x}$-axis, we have

$$s(t;\epsilon_1,\epsilon_2,\epsilon_3)\mu_{\bar{x}} = \vec{u} \tag{2.02}$$

with $s(t;\epsilon_1,\epsilon_2,\epsilon_3)$ being the speed of the ship on its course. Further we introduce the angular velocity vector $\vec{\omega}$ of the moving coordinate system as

$$\vec{\omega}(t;\epsilon_1,\epsilon_2,\epsilon_3) = \omega(t;\epsilon_1,\epsilon_2,\epsilon_3)\mu_z\,. \tag{2.03}$$

Under the assumption that the ship moved originally $(t \to -\infty)$ rectilinear in the direction of the x-axis, the angle $\alpha(t;\epsilon_1,\epsilon_2,\epsilon_3)$ between the vectors $\mu_{\bar{x}}$ and μ_x is given by

$$\alpha(t;\epsilon_1,\epsilon_2,\epsilon_3) = \int_{-\infty}^{t} \omega(t;\epsilon_1,\epsilon_2,\epsilon_3)\,dt \tag{2.04}$$

Now the transformation formulas from one coordinate system to the other are

$$x = \bar{x} \cdot \cos\alpha(t;\epsilon_1,\epsilon_2,\epsilon_3) + \bar{y}\sin\alpha(t;\epsilon_1,\epsilon_2,\epsilon_3) + x_c(t;\epsilon_1,\epsilon_2,\epsilon_3)$$

$$y = -\bar{x}\cdot\sin\alpha(t;\epsilon_1,\epsilon_2,\epsilon_3) + \bar{y}\cdot\cos\alpha(t;\epsilon_1,\epsilon_2,\epsilon_3) + y_c(t;\epsilon_1,\epsilon_2,\epsilon_3)$$

$$z = \bar{z}$$

$$\bar{x} = (x - x_c(t;\epsilon_1,\epsilon_2,\epsilon_3))\cos\alpha(t;\epsilon_1,\epsilon_2,\epsilon_3) - (y - y_c(t;\epsilon_1,\epsilon_2,\epsilon_3))\sin\alpha(t;\epsilon_1,\epsilon_2,\epsilon_3)$$

$$\bar{y} = (x - x_c(t;\epsilon_1,\epsilon_2,\epsilon_3))\sin\alpha(t;\epsilon_1,\epsilon_2,\epsilon_3) + (y - y_c(t;\epsilon_1,\epsilon_2,\epsilon_3))\cos\alpha(t;\epsilon_1,\epsilon_2,\epsilon_3)$$

$$\bar{z} = z \qquad (2.05)$$

The unit vectors along the axis of the coordinate systems transform according to

$$n_x = n_{\bar{x}}\cdot\cos\alpha(t;\epsilon_1,\epsilon_2,\epsilon_3) + n_{\bar{y}}\sin\alpha(t;\epsilon_1,\epsilon_2,\epsilon_3)$$

$$n_y = -n_{\bar{x}}\sin\alpha(t;\epsilon_1,\epsilon_2,\epsilon_3) + n_{\bar{y}}\cos\alpha(t;\epsilon_1,\epsilon_2,\epsilon_3)$$

$$n_z = n_{\bar{z}}$$

$$n_{\bar{x}} = n_x\cos\alpha(t;\epsilon_1,\epsilon_2,\epsilon_3) - n_y\sin\alpha(t;\epsilon_1,\epsilon_2,\epsilon_3)$$

$$n_{\bar{y}} = n_x\sin\alpha(t;\epsilon_1,\epsilon_2,\epsilon_3) + n_y\cdot\cos\alpha(t;\epsilon_1,\epsilon_2,\epsilon_3)$$

$$n_{\bar{z}} = n_z \qquad (2.06)$$

The equations (2.1) and (2.2) can now be written in the form

$$s(t;\epsilon_1,\epsilon_2,\epsilon_3)\, n_{\bar{x}} = \dot{x}_c(t;\epsilon_1,\epsilon_2,\epsilon_3)\, n_x + \dot{y}_c(t;\epsilon_1,\epsilon_2,\epsilon_3)\, n_y$$

and the substitution of the transformation (2.6) yields:

$$s(t;\epsilon_1,\epsilon_2,\epsilon_3)\, n_{\bar{x}} = \{\dot{x}_c(t;\epsilon_1,\epsilon_2,\epsilon_3)\cos\alpha(t;\epsilon_1,\epsilon_2,\epsilon_3) - \dot{y}_c(t;\epsilon_1,\epsilon_2,\epsilon_3)\sin\alpha(t;\epsilon_1,\epsilon_2,\epsilon_3)\}\, n_{\bar{x}}$$
$$+ \{\dot{x}_c(t;\epsilon_1,\epsilon_2,\epsilon_3)\sin\alpha(t;\epsilon_1,\epsilon_2,\epsilon_3) + \dot{y}_c(t;\epsilon_1,\epsilon_2,\epsilon_3)\cos\alpha(t;\epsilon_1,\epsilon_2,\epsilon_3)\}\, n_{\bar{y}}\,.$$

Therefore we have

$$\dot{x}_c(t;\epsilon_1,\epsilon_2,\epsilon_3)\cos\alpha(t;\epsilon_1,\epsilon_2,\epsilon_3) - \dot{y}_c(t;\epsilon_1,\epsilon_2,\epsilon_3)\sin\alpha(t;\epsilon_1,\epsilon_2,\epsilon_3) = s(t;\epsilon_1,\epsilon_2,\epsilon_3) \qquad (2.07)$$

$$\dot{x}_c(t;\epsilon_1,\epsilon_2,\epsilon_3)\sin\alpha(t;\epsilon_1,\epsilon_2,\epsilon_3) + \dot{y}_c(t;\epsilon_1,\epsilon_2,\epsilon_3)\cos\alpha(t;\epsilon_1,\epsilon_2,\epsilon_3) = 0 \qquad (2.08)$$

We will now proceed in giving the power series in terms of $\epsilon_1,\epsilon_2,\epsilon_3$ for our transformations. If

$$x_c(t;\epsilon_1,\epsilon_2,\epsilon_3) = x_{c000}(t) + \epsilon_1 x_{c100}(t) + \epsilon_1\epsilon_2 x_{c110}(t) + \epsilon_1\epsilon_3 x_{c101}(t) + \epsilon_1\epsilon_2\epsilon_3 x_{c111}(t) + \cdots \quad (2.09)$$

$$y_c(t;\epsilon_1,\epsilon_2,\epsilon_3) = y_{c000}(t) + \epsilon_1 y_{c100}(t) + \epsilon_1\epsilon_2 y_{c110}(t) + \epsilon_1\epsilon_3 y_{c101}(t) + \epsilon_1\epsilon_2\epsilon_3 y_{c111}(t) + \cdots \quad (2.10)$$

$$s(t;\epsilon_1,\epsilon_2,\epsilon_3) = s_{000}(t) + \epsilon_1 s_{100}(t) + \epsilon_1\epsilon_2 s_{110}(t) + \epsilon_1\epsilon_3 s_{101}(t) + \epsilon_1\epsilon_2\epsilon_3 s_{111}(t) + \cdots \quad (2.11)$$

$$\omega(t;\epsilon_1,\epsilon_2,\epsilon_3) = \omega_{000}(t) + \epsilon_1 \omega_{100}(t) + \epsilon_1\epsilon_2 \omega_{110}(t) + \epsilon_1\epsilon_3 \omega_{101}(t) + \epsilon_1\epsilon_2\epsilon_3 \omega_{111}(t) + \cdots \quad (2.12)$$

are the expansions in terms of ϵ_1, ϵ_2, ϵ_3 for the functions $x_c(t;\epsilon_1,\epsilon_2,\epsilon_3)$, $y_c(t;\epsilon_1,\epsilon_2,\epsilon_3)$, $s(t;\epsilon_1,\epsilon_2,\epsilon_3)$, $\omega(t;\epsilon_1,\epsilon_2,\epsilon_3)$. We next have from (2.4)

$$d(t;\epsilon_1,\epsilon_2,\epsilon_3) = d_{000}(t) + \epsilon_1 d_{100}(t) + \epsilon_1\epsilon_2 d_{110}(t) + \epsilon_1\epsilon_3 d_{101}(t) + \epsilon_1\epsilon_2\epsilon_3 d_{111}(t) + \cdots \quad (2.13)$$

with

$$d_{000}(t) = \int_{-\infty}^{t} \omega_{000}(t)\,dt \;,\quad d_{1jk}(t) = \int_{-\infty}^{t} \omega_{1jk}(t)\,dt \quad j,k = 0,1,\ldots \quad (2.14)$$

With (2.13) now follows:

$$\begin{aligned}
\cos d(t;\epsilon_1,\epsilon_2,\epsilon_3) &= \cos\{d_{000}(t) + \epsilon_1 d_{100}(t) + \epsilon_1\epsilon_2 d_{110}(t) + \epsilon_1\epsilon_3 d_{101}(t) + \epsilon_1\epsilon_2\epsilon_3 d_{111}(t) + \cdots\} \\
&= \cos d_{000}(t) \cdot \cos\{\epsilon_1 d_{100}(t) + \epsilon_1\epsilon_2 d_{110}(t) + \epsilon_1\epsilon_3 d_{101}(t) + \epsilon_1\epsilon_2\epsilon_3 d_{111}(t) + \cdots\} \\
&\quad - \sin d_{000}(t) \cdot \sin\{\epsilon_1 d_{100}(t) + \epsilon_1\epsilon_2 d_{110}(t) + \epsilon_1\epsilon_3 d_{101}(t) + \epsilon_1\epsilon_2\epsilon_3 d_{111}(t) + \cdots\}
\end{aligned}$$

$$\begin{aligned}
\sin d(t;\epsilon_1,\epsilon_2,\epsilon_3) &= \sin\{d_{000}(t) + \epsilon_1 d_{100}(t) + \epsilon_1\epsilon_2 d_{110}(t) + \epsilon_1\epsilon_3 d_{101}(t) + \epsilon_1\epsilon_2\epsilon_3 d_{111}(t) + \cdots\} \\
&= \sin d_{000}(t) \cdot \cos\{\epsilon_1 d_{100}(t) + \epsilon_1\epsilon_2 d_{110}(t) + \epsilon_1\epsilon_3 d_{101}(t) + \epsilon_1\epsilon_2\epsilon_3 d_{111}(t) + \cdots\} \\
&\quad + \cos d_{000}(t) \cdot \sin\{\epsilon_1 d_{100}(t) + \epsilon_1\epsilon_2 d_{110}(t) + \epsilon_1\epsilon_3 d_{101}(t) + \epsilon_1\epsilon_2\epsilon_3 d_{111}(t) + \cdots\}
\end{aligned}$$

Furthermore there are the expansions:

$$\cos\alpha(t;\varepsilon_1,\varepsilon_2,\varepsilon_3) = \cos\alpha_{000}(t) - \varepsilon_1\alpha_{100}(t)\cdot\sin\alpha_{000}(t) - \varepsilon_1\varepsilon_2\alpha_{110}(t)\cdot\sin\alpha_{000}(t)$$
$$- \varepsilon_1\varepsilon_3\alpha_{101}(t)\cdot\sin\alpha_{000}(t) - \varepsilon_1\varepsilon_2\varepsilon_3\alpha_{111}(t)\cdot\sin\alpha_{000}(t) + \cdots$$

$$\sin\alpha(t;\varepsilon_1,\varepsilon_2,\varepsilon_3) = \sin\alpha_{000}(t) + \varepsilon_1\alpha_{100}(t)\cdot\cos\alpha_{000}(t) + \varepsilon_1\varepsilon_2\alpha_{110}(t)\cdot\cos\alpha_{000}(t)$$
$$+ \varepsilon_1\varepsilon_3\alpha_{101}(t)\cos\alpha_{000}(t) + \varepsilon_1\varepsilon_2\varepsilon_3\alpha_{111}(t)\cdot\cos\alpha_{000}(t) + \cdots$$

If we substitute the expansions (2.9) - (2.13) into the transformation formulas (2.5) and (2.6) we get:

$$x = \bar{x}\cdot\cos\alpha_{000}(t) + \bar{y}\cdot\sin\alpha_{000}(t) + x_{c000}(t) + \varepsilon_1\sum_{j,k=0}^{1}\varepsilon_2^j\varepsilon_3^k\{\alpha_{1jk}(t)(-\bar{x}\sin\alpha_{000}(t) + \bar{y}\cos\alpha_{000}(t)) + x_{c1jk}(t)\} + \cdots$$

$$y = -\bar{x}\sin\alpha_{000}(t) + \bar{y}\cos\alpha_{000}(t) + y_{c000}(t) - \varepsilon_1\sum_{j,k=0}^{1}\varepsilon_2^j\varepsilon_3^k\{\alpha_{1jk}(t)(\bar{x}\cos\alpha_{000}(t) + \bar{y}\sin\alpha_{000}(t)) + y_{c1jk}(t)\} + \cdots$$

$$z = \bar{z} \qquad\qquad (2.15)$$

$$\bar{x} = (x - x_{c000}(t))\cos\alpha_{000}(t) - (y - y_{c000}(t))\sin\alpha_{000}(t)$$
$$+ \varepsilon_1\sum_{j,k=0}^{1}\varepsilon_2^j\varepsilon_3^k\{-\alpha_{1jk}(t)[(x - x_{c000}(t))\sin\alpha_{000}(t) + (y - y_{c000}(t))\cos\alpha_{000}(t)]$$
$$- x_{c1jk}(t)\cdot\cos\alpha_{000}(t) + y_{c1jk}(t)\sin\alpha_{000}(t)\} + \cdots$$

$$\bar{y} = (x - x_{c000}(t))\sin\alpha_{000}(t) + (y - y_{c000}(t))\cos\alpha_{000}(t)$$
$$+ \varepsilon_1\sum_{j,k=0}^{1}\varepsilon_2^j\varepsilon_3^k\{\alpha_{1jk}(t)[(x - x_{c000}(t))\cos\alpha_{000}(t) - (y - y_{c000}(t))\sin\alpha_{000}(t)]$$
$$- x_{c1jk}(t)\sin\alpha_{000}(t) - y_{c1jk}(t)\cos\alpha_{000}(t)\} + \cdots$$

$$\bar{z} = z \qquad\qquad (2.16)$$

and

$$u_x = u_{\bar{x}}\cos\alpha_{000}(t) + u_{\bar{y}}\sin\alpha_{000}(t) + \varepsilon_1\sum_{j,k=0}^{1}\varepsilon_2^j\varepsilon_3^k\{-u_{\bar{x}}\cdot\sin\alpha_{000}(t) + u_{\bar{y}}\cos\alpha_{000}(t)\}\alpha_{1jk}(t) + \cdots$$

$$u_y = -u_{\bar{x}}\sin\alpha_{000}(t) + u_{\bar{y}}\cos\alpha_{000}(t) - \varepsilon_1\sum_{j,k=0}^{1}\varepsilon_2^j\varepsilon_3^k\{u_{\bar{x}}\cos\alpha_{000}(t) + u_{\bar{y}}\sin\alpha_{000}(t)\}\alpha_{1jk}(t) + \cdots$$

$$u_z = u_{\bar{z}} \qquad\qquad (2.17)$$

$$u_{\bar{x}} = u_x\cdot\cos\alpha_{000}(t) - u_y\sin\alpha_{000}(t) - \varepsilon_1\sum_{j,k=0}^{1}\varepsilon_2^j\varepsilon_3^k\{u_x\cdot\sin\alpha_{000}(t) + u_y\cos\alpha_{000}(t)\}\alpha_{1jk}(t) + \cdots$$

$$u_{\bar{y}} = u_x\cdot\sin\alpha_{000}(t) + u_y\cos\alpha_{000}(t) + \varepsilon_1\sum_{j,k=0}^{1}\varepsilon_2^j\varepsilon_3^k\{u_x\cdot\cos\alpha_{000}(t) - u_y\sin\alpha_{000}(t)\}\alpha_{1jk}(t) + \cdots$$

$$u_{\bar{z}} = u_z \qquad\qquad (2.18)$$

For the derivation of the boundary conditions on the ship's hull it is convenient, to introduce another moving coordinate system with axis $\bar x'$, $\bar y'$, $\bar z'$ which is rigidly attached to the ship. It is assumed that the hull of the ship has a vertical plane of symmetry and that this plane is also the plane of symmetry for the mass distribution of the ship. We now locate the $\bar x'$, $\bar z'$-plane in this vertical plane of symmetry and suppose that the $\bar z'$-axis contains the center of gravity (Figure 2). In the at rest position of equilibrium the $\bar x'$, $\bar y'$, $\bar z'$-system and the $\bar x$, $\bar y$, $\bar z$-system may coincide. With respect to the $\bar x'$, $\bar y'$, $\bar z'$-system the center of gravity has the coordinates $(0, 0, \bar z_c')$.

Because in this paper we only deal with motions of the ship which deviate only little from the translation of the ship with constant velocity in the direction of the x-axis, it is convenient to suppose that the angular displacement of the $\bar x'$, $\bar y'$, $\bar z'$-system relative to the $\bar x$, $\bar y$, $\bar z$-system is so small that it can be treated as a vector:

$$\vartheta(t;\epsilon_1,\epsilon_2,\epsilon_3) = \mathbf{i}_{\bar x'}\Theta_1(t;\epsilon_1,\epsilon_2,\epsilon_3) + \mathbf{i}_{\bar y'}\Theta_2(t;\epsilon_1,\epsilon_2,\epsilon_3) + \mathbf{i}_{\bar z'}\Theta_3(t;\epsilon_1,\epsilon_2,\epsilon_3) \tag{2.19}$$

With $\bar z_c(t;\epsilon_1,\epsilon_2,\epsilon_3)$ as $\bar z$-coordinate of the center of gravity we have the following transformation formulas correct up to first order terms in the components $\Theta_i(t;\epsilon_1,\epsilon_2,\epsilon_3)$ $i = 1, 2, 3$ of $\vartheta(t;\epsilon_1,\epsilon_2,\epsilon_3)$:

$$\bar{\mathbf{r}}' - (0,0,\bar z_c') = \bar{\mathbf{r}} - (0,0,\bar z_c(t;\epsilon_1,\epsilon_2,\epsilon_3)) + \vartheta(t;\epsilon_1,\epsilon_2,\epsilon_3)\times\{\bar{\mathbf{r}} - (0,0,\bar z_c(t;\epsilon_1,\epsilon_2,\epsilon_3))\}$$

or

$$\bar{\mathbf{r}} - (0,0,\bar z_c(t;\epsilon_1,\epsilon_2,\epsilon_3)) = \bar{\mathbf{r}}' - (0,0,\bar z_c') - \vartheta(t;\epsilon_1,\epsilon_2,\epsilon_3)\times\{\bar{\mathbf{r}}' - (0,0,\bar z_c')\}.$$

The components of these transformation formulas are:

$$\bar x' = \bar x - \Theta_3(t;\epsilon_1,\epsilon_2,\epsilon_3)\bar y + \Theta_2(t;\epsilon_1,\epsilon_2,\epsilon_3)(\bar z - \bar z_c(t;\epsilon_1,\epsilon_2,\epsilon_3))$$

$$\bar y' = \bar y - \Theta_1(t;\epsilon_1,\epsilon_2,\epsilon_3)(\bar z - \bar z_c(t;\epsilon_1,\epsilon_2,\epsilon_3)) + \Theta_3(t;\epsilon_1,\epsilon_2,\epsilon_3)\bar x$$

$$\bar z = \bar z - (\bar z_c(t;\epsilon_1,\epsilon_2,\epsilon_2) - \bar z_c') - \Theta_2(t;\epsilon_1,\epsilon_2,\epsilon_3)\bar x + \Theta_1(t;\epsilon_1,\epsilon_2,\epsilon_3)\bar y \tag{2.20}$$

and

$$\bar{x} = \bar{x}' + \Theta_3(t;\epsilon_1,\epsilon_2,\epsilon_3)\bar{y}' - \Theta_2(t;\epsilon_1,\epsilon_2,\epsilon_3)(\bar{z}'-\bar{z}_c')$$

$$\bar{y} = \bar{y}' + \Theta_1(t;\epsilon_1,\epsilon_2,\epsilon_3)(\bar{z}'-\bar{z}_c') - \Theta_3(t;\epsilon_1,\epsilon_2,\epsilon_3)\bar{x}'$$

$$\bar{z} = \bar{z}' - (\bar{z}_c' - \bar{z}_c(t;\epsilon_1,\epsilon_2,\epsilon_3)) + \Theta_2(t;\epsilon_1,\epsilon_2,\epsilon_3)\bar{x}' - \Theta_1(t;\epsilon_1,\epsilon_2,\epsilon_3)\bar{y}' \quad (2.21)$$

Now we suppose the following developments in terms of $\epsilon_1,\epsilon_2,\epsilon_3$

$$\Theta_i(t;\epsilon_1,\epsilon_2,\epsilon_3) = \epsilon_1\Theta_{i100}(t) + \epsilon_1\epsilon_2\Theta_{i110}(t) + \epsilon_1\epsilon_3\Theta_{i101}(t) + \epsilon_1\epsilon_2\epsilon_3\Theta_{i111}(t) + \cdots \qquad i=1,2,3 \quad (2.22)$$

$$\bar{z}_c(t;\epsilon_1,\epsilon_2,\epsilon_3) - \bar{z}_c' = \epsilon_1 z_{100}(t) + \epsilon_1\epsilon_2 z_{110}(t) + \epsilon_1\epsilon_3 z_{101}(t) + \epsilon_1\epsilon_2\epsilon_3 z_{111}(t) + \cdots \quad (2.23)$$

All these developments state, that the motion of the ship with respect to the $\bar{x}$, $\bar{y}$, $\bar{z}$-coordinate system is small of order ϵ_1. If we now substitute the expansions (2.22), and (2.23) into the transformation formulas (2.20) and (2.21) there results:

$$\bar{x}' = \bar{x} + \epsilon_1\sum_{j,k=0}^{1}\epsilon_2^j\epsilon_3^k\{-\Theta_{31jk}(t)\bar{y} + \Theta_{21jk}(t)(\bar{z}-\bar{z}_c')\} + \cdots$$

$$\bar{y}' = \bar{y} + \epsilon_1\sum_{j,k=0}^{1}\epsilon_2^j\epsilon_3^k\{-\Theta_{11jk}(t)(\bar{z}-\bar{z}_c') + \Theta_{31jk}(t)\bar{x}\} + \cdots$$

$$\bar{z}' = \bar{z} - \epsilon_1\sum_{j,k=0}^{1}\epsilon_2^j\epsilon_3^k\{z_{1jk}(t) + \Theta_{21jk}(t)\bar{x} - \Theta_{11jk}(t)\bar{y}\} + \cdots \quad (2.24)$$

and

$$\bar{x} = \bar{x}' + \epsilon_1\sum_{j,k=0}^{1}\epsilon_2^j\epsilon_3^k\{\Theta_{31jk}(t)\bar{y}' - \Theta_{21jk}(t)(\bar{z}'-\bar{z}_c')\} + \cdots$$

$$\bar{y} = \bar{y}' + \epsilon_1\sum_{j,k=0}^{1}\epsilon_2^j\epsilon_3^k\{\Theta_{11jk}(t)(\bar{z}'-\bar{z}_c') - \Theta_{31jk}(t)\bar{x}'\} + \cdots$$

$$\bar{z} = \bar{z}' + \epsilon_1\sum_{j,k=0}^{1}\epsilon_2^j\epsilon_3^k\{z_{1jk}(t) + \Theta_{21jk}(t)\bar{x}' - \Theta_{11jk}(t)\bar{y}'\} + \cdots \quad (2.25)$$

which are correct up to first order terms in ϵ_1. The appropriate formulas for the transformation of the unit vectors, correct up to first order terms in ϵ_1 are:

$$n_{\bar{x}'} = n_{\bar{x}} + \epsilon_1 \sum_{j,k=0}^{1} \epsilon_2^j \epsilon_3^k \{ -\Theta_{31jk}(t) n_{\bar{y}} + \Theta_{21jk}(t) n_{\bar{z}} \} + \cdots$$

$$n_{\bar{y}'} = n_{\bar{y}} + \epsilon_1 \sum_{j,k=0}^{1} \epsilon_2^j \epsilon_3^k \{ -\Theta_{11jk}(t) n_{\bar{z}} + \Theta_{31jk}(t) n_{\bar{x}} \} + \cdots$$

$$n_{\bar{z}'} = n_{\bar{z}} + \epsilon_1 \sum_{j,k=0}^{1} \epsilon_2^j \epsilon_3^k \{ -\Theta_{21jk}(t) n_{\bar{x}} + \Theta_{11jk}(t) n_{\bar{y}} \} + \cdots \quad (2.26)$$

$$n_{\bar{x}} = n_{\bar{x}'} + \epsilon_1 \sum_{j,k=0}^{1} \epsilon_2^j \epsilon_3^k \{ \Theta_{31jk}(t) n_{\bar{y}'} - \Theta_{21jk}(t) n_{\bar{z}'} \} + \cdots$$

$$n_{\bar{y}} = n_{\bar{y}'} + \epsilon_1 \sum_{j,k=0}^{1} \epsilon_2^j \epsilon_3^k \{ \Theta_{11jk}(t) n_{\bar{z}'} - \Theta_{31jk}(t) n_{\bar{x}'} \} + \cdots$$

$$n_{\bar{z}} = n_{\bar{z}'} + \epsilon_1 \sum_{j,k=0}^{1} \epsilon_2^j \epsilon_3^k \{ \Theta_{21jk}(t) n_{\bar{x}'} - \Theta_{11jk}(t) n_{\bar{y}'} \} + \cdots \quad (2.27)$$

If we now substitute the transformation formula (2.16) into formula (2.24) we arrive at the formulas for the transformation from the $\bar{x}'$, $\bar{y}'$, $\bar{z}'$-coordinate system, rigidly attached to the ship, to the fixed x, y, z-coordinate system:

$$\begin{aligned}\bar{x}' = {} & (x - x_{c000}(t)) \cos \alpha_{000}(t) - (y - y_{c000}(t)) \sin \alpha_{000}(t) \\ & + \epsilon_1 \sum_{j,k=0}^{1} \epsilon_2^j \epsilon_3^k \{ -(x - x_{c000}(t)) \{\Theta_{31jk}(t) + \alpha_{1jk}(t)\} \sin \alpha_{000}(t) \\ & \qquad - (y - y_{c000}(t)) \{\Theta_{31jk}(t) + \alpha_{1jk}(t)\} \cos \alpha_{000}(t) \\ & \qquad - x_{c1jk}(t) \cos \alpha_{000}(t) + y_{c1jk}(t) \sin \alpha_{000}(t) + \Theta_{21jk}(t)(z - \bar{z}_c') \} + \cdots\end{aligned}$$

$$\begin{aligned}\bar{y}' = {} & (x - x_{c000}(t)) \sin \alpha_{000}(t) + (y - y_{c000}(t)) \cos \alpha_{000}(t) \\ & + \epsilon_1 \sum_{j,k=0}^{1} \epsilon_2^j \epsilon_3^k \{ (x - x_{c000}(t)) \{\Theta_{31jk}(t) + \alpha_{1jk}(t)\} \cos \alpha_{000}(t) \\ & \qquad - (y - y_{c000}(t)) \{\Theta_{31jk}(t) + \alpha_{1jk}(t)\} \sin \alpha_{000}(t) \\ & \qquad - x_{c1jk}(t) \sin \alpha_{000}(t) - y_{c1jk}(t) \cos \alpha_{000}(t) - \Theta_{11jk}(t)(z - \bar{z}_c') \} + \cdots \quad (2.28)\end{aligned}$$

$$\begin{aligned}\bar{z}' = {} & z + \epsilon_1 \sum_{j,k=0}^{1} \epsilon_2^j \epsilon_3^k \{ -z_{1jk}(t) + (x - x_{c000}(t)) \{\Theta_{21jk}(t) \cos \alpha_{000}(t) - \Theta_{11jk}(t) \sin \alpha_{000}(t)\} \\ & \qquad + (y - y_{c000}(t)) \{\Theta_{21jk}(t) \sin \alpha_{000}(t) + \Theta_{11jk}(t) \cos \alpha_{000}(t)\} \} + \cdots\end{aligned}$$

Substitution of formula (2.25) in formula (2.15) gives for the inverse transformation:

$$
\begin{aligned}
x = {} & \bar{x}'\cdot\cos\alpha_{000}(t) + \bar{y}'\sin\alpha_{000}(t) + x_{c000}(t) \\
& + \varepsilon_1 \sum_{j,k=0}^{1} \varepsilon_2^j \varepsilon_3^k \Big\{ -\bar{x}'\{\Theta_{31jk}(t)+\alpha_{1jk}(t)\}\sin\alpha_{000}(t) + \bar{y}'\{\Theta_{31jk}(t)+\alpha_{1jk}(t)\}\cos\alpha_{000}(t) \\
& \qquad + x_{c1jk}(t) + (\bar{z}'-\bar{z}_c')\{\Theta_{21jk}(t)\cos\alpha_{000}(t) - \Theta_{11jk}(t)\sin\alpha_{000}(t)\}\Big\} + \cdots \\
y = {} & -\bar{x}'\sin\alpha_{000}(t) + \bar{y}'\cos\alpha_{000}(t) + y_{c000}(t) \\
& + \varepsilon_1 \sum_{j,k=0}^{1} \varepsilon_2^j \varepsilon_3^k \Big\{ -\bar{x}'\{\Theta_{31jk}(t)+\alpha_{1jk}(t)\}\cos\alpha_{000}(t) - \bar{y}'\{\Theta_{31jk}(t)+\alpha_{1jk}(t)\}\sin\alpha_{000}(t) \\
& \qquad + y_{c1jk}(t) + (\bar{z}'-\bar{z}_c')\{\Theta_{21jk}(t)\sin\alpha_{000}(t) + \Theta_{11jk}(t)\cos\alpha_{000}(t)\}\Big\} + \cdots \\
z = {} & \bar{z}' + \varepsilon_1 \sum_{j,k=0}^{1} \varepsilon_2^j \varepsilon_3^k \{ z_{1jk}(t) + \bar{x}'\,\Theta_{21jk}(t) - \bar{y}'\,\Theta_{11jk}(t)\} + \cdots
\end{aligned}
\tag{2.29}
$$

For the unit vectors in the direction of the axes of the coordinate systems we have the transformation formulas:

$$
\begin{aligned}
n_{\bar{x}'} = {} & n_x\cdot\cos\alpha_{000}(t) - n_y\sin\alpha_{000}(t) \\
& + \varepsilon_1 \sum_{j,k=0}^{1} \varepsilon_2^j \varepsilon_3^k \Big\{ -n_x\{\Theta_{31jk}(t)+\alpha_{1jk}(t)\}\sin\alpha_{000}(t) \\
& \qquad - n_y\{\Theta_{31jk}(t)+\alpha_{1jk}(t)\}\cos\alpha_{000}(t) + n_z\Theta_{21jk}(t)\Big\} + \cdots \\
n_{\bar{y}'} = {} & n_x\cdot\sin\alpha_{000}(t) + n_y\cos\alpha_{000}(t) \\
& + \varepsilon_1 \sum_{j,k=0}^{1} \varepsilon_2^j \varepsilon_3^k \Big\{ n_x\{\Theta_{31jk}(t)+\alpha_{1jk}(t)\}\cos\alpha_{000}(t) \\
& \qquad - n_y\{\Theta_{31jk}(t)+\alpha_{1jk}(t)\}\sin\alpha_{000}(t) - n_z\Theta_{11jk}(t)\Big\} + \cdots \\
n_{\bar{z}'} = {} & n_z + \varepsilon_1 \sum_{j,k=0}^{1} \varepsilon_2^j \varepsilon_3^k \Big\{ -n_x\{\Theta_{21jk}(t)\cos\alpha_{000}(t) - \Theta_{11jk}(t)\sin\alpha_{000}(t)\} \\
& \qquad + n_y\{\Theta_{21jk}(t)\sin\alpha_{000}(t) + \Theta_{11jk}(t)\cos\alpha_{000}(t)\}\Big\} + \cdots
\end{aligned}
\tag{2.30}
$$

and

$$
\begin{aligned}
n_x = \; & n_{\bar{x}'} \cdot \cos \alpha_{000}(t) + n_{\bar{y}'} \sin \alpha_{000}(t) \\
& + \epsilon_1 \sum_{j,k=0}^{1} \epsilon_2^j \epsilon_3^k \Big\{ - n_{\bar{x}'} \{ \Theta_{31jk}(t) + \alpha_{1jk}(t) \} \sin \alpha_{000}(t) \\
& \qquad + n_{\bar{y}'} \{ \Theta_{31jk}(t) + \alpha_{1jk}(t) \} \cos \alpha_{000}(t) \\
& \qquad - n_{\bar{z}'} \{ \Theta_{21jk}(t) \cos \alpha_{000}(t) - \Theta_{11jk}(t) \sin \alpha_{000}(t) \} \Big\} + \cdots
\end{aligned}
$$

$$
\begin{aligned}
n_y = \; & - n_{\bar{x}'} \sin \alpha_{000}(t) + n_{\bar{y}'} \cos \alpha_{000}(t) \\
& + \epsilon_1 \sum_{j,k=0}^{1} \epsilon_2^j \epsilon_3^k \Big\{ - n_{\bar{x}'} \{ \Theta_{31jk}(t) + \alpha_{1jk}(t) \} \cos \alpha_{000}(t) \\
& \qquad - n_{\bar{y}'} \{ \Theta_{31jk}(t) + \alpha_{1jk}(t) \} \sin \alpha_{000}(t) \\
& \qquad + n_{\bar{z}} \{ \Theta_{21jk}(t) \sin \alpha_{000}(t) + \Theta_{11jk}(t) \cos \alpha_{000}(t) \} \Big\} + \cdots
\end{aligned}
\tag{2.31}
$$

$$
n_z = n_{\bar{z}'} + \epsilon_1 \sum_{j,k=0}^{1} \epsilon_2^j \epsilon_3^k \{ n_{\bar{x}'} \Theta_{21jk}(t) - n_{\bar{y}'} \Theta_{11jk}(t) \} + \cdots
$$

When the boundary conditions on the hull of the ship are derived we shall prove that

$$\dot{\alpha}_{000}(t) = 0 \tag{2.32}$$

Thus

$$\alpha_{000}(t) \equiv \text{const.} \tag{2.33}$$

Because we are interested in the motion of a ship under the action of a constant propeller thrust and rudder force in a canal of variable breadth and depth, we may assume the motion of the ship to be in the direction of the x-axis for $t \to -\infty$. Therefore from (2.33) we get

$$\alpha_{000}(t) \equiv 0 \; . \tag{2.34}$$

If we now substitute the power series (2.11) and (2.13) in equations (2.7) and (2.8) the terms of zero order in ϵ_1, ϵ_2, ϵ_3 give the conditions

$$\dot{x}_{c000}(t) \cdot \cos \alpha_{000}(t) - \dot{y}_{c000}(t) \sin \alpha_{000}(t) = s_{000}(t) \tag{2.35}$$

$$\dot{x}_{c000}(t) \sin \alpha_{000}(t) + \dot{y}_{c000}(t) \cos \alpha_{000}(t) = 0 \tag{2.36}$$

On account of (2.34) we find the equations:

$$\dot{x}_{c000}(t) = s_{000}(t) \tag{2.37}$$

$$\dot{y}_{c000}(t) = 0 \quad . \tag{2.38}$$

Therefore we yield the following coordinates of zero order for the center of gravity of the ship with respect to the fixed x,y,z-coordinates system:

$$x_{c000}(t) = \int_{t_0}^{t} s_{000}(t)\,dt \tag{2.39}$$

$$y_{c000}(t) = 0 \quad . \tag{2.40}$$

Here we have assumed $x_{c000}(t_0) = 0$. Also we used our assumption about the motion of the ship for $t \to -\infty$. For future reference the following formulas may be noticed

$$\begin{aligned} &\dot{x}_{c1jk}(t)\cos\alpha_{000}(t) - \dot{x}_{c000}(t)\cdot\alpha_{1jk}(t)\sin\alpha_{000}(t) \\ &- \dot{y}_{c1jk}(t)\sin\alpha_{000}(t) - \dot{y}_{c000}(t)\cdot\alpha_{1jk}(t)\cos\alpha_{000}(t) = s_{1jk}(t) \end{aligned} \tag{2.41}$$

$$j,k = 1,2$$

$$\begin{aligned} &\dot{x}_{c1jk}(t)\sin\alpha_{000}(t) + \dot{x}_{c000}(t)\,\alpha_{1jk}(t)\cos\alpha_{000}(t) \\ &+ \dot{y}_{c1jk}(t)\cos\alpha_{000}(t) - \dot{y}_{c000}(t)\,\alpha_{1jk}(t)\sin\alpha_{000}(t) = 0 \quad , \end{aligned} \tag{2.42}$$

which are derived from (2.7) and (2.8) by considering terms of order ϵ_1, $\epsilon_1 \cdot \epsilon_2$, $\epsilon_1 \cdot \epsilon_3$ and $\epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3$. With $\alpha_{000}(t) \equiv 0$ the following relations result:

$$\dot{x}_{c1jk}(t) - \dot{y}_{c000}(t)\cdot\alpha_{1jk}(t) = s_{1jk}(t)\text{, or on account of (2.38): } \dot{x}_{c1jk}(t) = s_{1jk}(t) \tag{2.43}$$

$$j,k = 0,1$$

$$\dot{x}_{c000}(t)\cdot\alpha_{1jk}(t) + \dot{y}_{c1jk}(t) = 0\text{, or on account of (2.37): } s_{000}(t)\,\alpha_{1jk}(t) + \dot{y}_{c1jk}(t) = 0 \quad . \tag{2.44}$$

Also for future reference we notice that the potential $\bar{\Phi}$ and dynamical pressure $\bar{P}$ with respect to the $\bar{x}$, $\bar{y}$, $\bar{z}$-coordinate system obey the following identities:

$$\bar{\phi}(\bar{x},\bar{y},\bar{z},t;\epsilon_1,\epsilon_2,\epsilon_3) \equiv \phi\big(\bar{x}\cdot\cos\alpha(t;\epsilon_1,\epsilon_2,\epsilon_3)+\bar{y}\sin\alpha(t;\epsilon_1,\epsilon_2,\epsilon_3)+x_c(t;\epsilon_1,\epsilon_2,\epsilon_3), \\ -\bar{x}\sin\alpha(t;\epsilon_1,\epsilon_2,\epsilon_3)+\bar{y}\cos\alpha(t;\epsilon_1,\epsilon_2,\epsilon_3)+y_c(t;\epsilon_1,\epsilon_2,\epsilon_3), \bar{z}, t; \epsilon_1,\epsilon_2,\epsilon_3\big) \tag{2.45}$$

$$\bar{p}(\bar{x},\bar{y},\bar{z},t;\epsilon_1,\epsilon_2,\epsilon_3) \equiv p\big(\bar{x}\cdot\cos\alpha(t;\epsilon_1,\epsilon_2,\epsilon_3)+\bar{y}\sin\alpha(t;\epsilon_1,\epsilon_2,\epsilon_3)+x_c(t;\epsilon_1,\epsilon_2,\epsilon_3), \\ -\bar{x}\sin\alpha(t;\epsilon_1,\epsilon_2,\epsilon_3)+\bar{y}\cos\alpha(t;\epsilon_1,\epsilon_2,\epsilon_3)+y_c(t;\epsilon_1,\epsilon_2,\epsilon_3), \bar{z}, t; \epsilon_1,\epsilon_2,\epsilon_3\big) \tag{2.46}$$

thus we have the following developments in powers of ϵ_1, ϵ_2, ϵ_3

$$\bar{\phi}(\bar{x},\bar{y},\bar{z},t;\epsilon_1,\epsilon_2,\epsilon_3) = \epsilon_1\bar{\phi}_{100}(\bar{x},\bar{y},\bar{z},t)+\epsilon_1\epsilon_2\bar{\phi}_{110}(\bar{x},\bar{y},\bar{z},t)+\epsilon_1\epsilon_3\bar{\phi}_{101}(\bar{x},\bar{y},\bar{z},t) \\ +\epsilon_1\epsilon_2\epsilon_3\bar{\phi}_{111}(\bar{x},\bar{y},\bar{z},t)+\cdots \tag{2.47}$$

$$\bar{p}(\bar{x},\bar{y},\bar{z},t;\epsilon_1,\epsilon_2,\epsilon_3) = \epsilon_1\bar{p}_{100}(\bar{x},\bar{y},\bar{z},t)+\epsilon_1\epsilon_2\bar{p}_{110}(\bar{x},\bar{y},\bar{z},t)+\epsilon_1\epsilon_3\bar{p}_{101}(\bar{x},\bar{y},\bar{z},t) \\ +\epsilon_1\epsilon_2\epsilon_3\bar{p}_{111}(\bar{x},\bar{y},\bar{z},t)+\cdots, \tag{2.48}$$

as can be seen from the transformation formulas (2.15) and the developments (1.13) and (1.16). In these formulas

$$\bar{\phi}_{1jk}(\bar{x},\bar{y},\bar{z},t) \equiv \phi_{1jk}\big(\bar{x}\cos\alpha_{000}(t)+\bar{y}\sin\alpha_{000}(t)+x_{c000}(t), \\ -\bar{x}\sin\alpha_{000}(t)+\bar{y}\cos\alpha_{000}(t)+y_{c000}(t), \bar{z}, t\big) \tag{2.49}$$

$$j,k=0,1$$

$$\bar{p}_{1jk}(\bar{x},\bar{y},\bar{z},t) \equiv p_{1jk}\big(\bar{x}\cos\alpha_{000}(t)+\bar{y}\sin\alpha_{000}(t)+x_{c000}(t), \\ -\bar{x}\sin\alpha_{000}(t)+\bar{y}\cos\alpha_{000}(t)+y_{c000}(t), \bar{z}, t\big) . \tag{2.50}$$

On account of (1.22) and (1.44) it now follows, that these functions satisfy the Laplace's equation:

$$\nabla^2\bar{\phi}_{1jk} = \bar{\phi}_{1jk\bar{x}\bar{x}}+\bar{\phi}_{1jk\bar{y}\bar{y}}+\bar{\phi}_{1jk\bar{z}\bar{z}} = 0 \tag{2.51}$$

$$\nabla^2\bar{p}_{1jk} = \bar{p}_{1jk\bar{x}\bar{x}}+\bar{p}_{1jk\bar{y}\bar{y}}+\bar{p}_{1jk\bar{z}\bar{z}} = 0 \tag{2.52}$$

$$\left\{\begin{array}{l} -h<\bar{z}<0 \\ -B_2<\bar{y}<B_1 \\ \text{outside the ship} \\ j,k=0,1 \end{array}\right.$$

3. The Boundary Condition on the Ships Surface

In our theory we assumed all the physical variables to be functions of the parameters ϵ_1, ϵ_2 and ϵ_3. The meaning of the parameters ϵ_2, ϵ_3 can be inferred from (1.32) and (1.38). The parameter ϵ_1 however should be the beam-length ratio of the ship, introduced in our theory by the equation of the ship's surface

$$\bar{y}' = \pm \epsilon_1 h(\bar{x}', \bar{z}') \qquad \bar{y}' \gtrless 0 \; ; \; \bar{x}', \bar{z}' \text{ in } \bar{A}' \tag{3.01}$$

with respect to the $\bar{x}'$, $\bar{y}'$, $\bar{z}'$-coordinate system rigidly attached to the ship (Figure 3). Here the function $h(\bar{x}', \bar{z}')$ is defined over a region $\bar{A}'$ of the $\bar{x}'$, $\bar{z}'$-plane (Figure 4). We see the ship's surface reduces to $\bar{A}'$ for $\epsilon_1 \to 0$ and a translation of such a surface parallel to the x,z-plane would give no surface waves on account of our assumption of an inviscid, incompressible and irrotational flow. Only for $\epsilon_1 \neq 0$ the translatory motion with constant (or variable) velocity parallel to the x-axis as also the superposed small perturbations would give surface waves of small amplitude, so that our assumptions with respect to the expansions of the physical variables are justified. Now we proceed in giving the boundary conditions on the ship's hull.

Let the ship's hull be given by a relation of the form

$$H(x, y, z, t; \epsilon_1, \epsilon_2, \epsilon_3) = 0 \tag{3.02}$$

with respect to the fixed x,y,z-coordinate system. The kinematic condition on the ship's hull is

$$\frac{dH}{dt} = \phi_x H_x + \phi_y H_y + \phi_z H_z + H_t = 0 \qquad \text{on } H(x, y, z, t; \epsilon_1, \epsilon_2, \epsilon_3) = 0. \tag{3.03}$$

Since the equation of the ship's hull (3.1) is firstly given with respect to the $\bar{x}'$,$\bar{y}'$,$\bar{z}'$-system, we have to transform the equation (3.1) with the help of the transformation formulas (2.28) to the function $H(x,y,z,t; \epsilon_1,\epsilon_2,\epsilon_3)$ which appears in the boundary condition (3.3). Let

$$\bar{H}(\bar{x}', \bar{y}', \bar{z}', t; \epsilon_1, \epsilon_2, \epsilon_3) \equiv \bar{y}' \mp \epsilon_1 h(\bar{x}', \bar{z}') = 0 \qquad \bar{y}' \gtrless 0 \quad \bar{x}', \bar{z}' \text{ in } \bar{A}'. \tag{3.04}$$

Then with regard to (2.28) we have

$$
\begin{aligned}
H(x,y,z,t;\epsilon_1,\epsilon_2,\epsilon_3) = {} & \bar{H}(\bar{x}',\bar{y}',\bar{z}',t;\epsilon_1,\epsilon_2,\epsilon_3)\Big|_{\epsilon_1=\epsilon_2=\epsilon_3=0} \\
& + \epsilon_1\left[\bar{H}_{\bar{x}'}\frac{\partial \bar{x}'}{\partial \epsilon_1} + \bar{H}_{\bar{y}'}\frac{\partial \bar{y}'}{\partial \epsilon_1} + \bar{H}_{\bar{z}'}\frac{\partial \bar{z}'}{\partial \epsilon_1} + H_{\epsilon_1}\right]_{\epsilon_1=\epsilon_2=\epsilon_3=0} \\
& + \epsilon_1\epsilon_2\left[\bar{H}_{\bar{x}'}\frac{\partial^2 \bar{x}'}{\partial \epsilon_1\partial \epsilon_2} + \bar{H}_{\bar{y}'}\frac{\partial^2 \bar{y}'}{\partial \epsilon_1\partial \epsilon_2} + \bar{H}_{\bar{z}'}\frac{\partial^2 \bar{z}'}{\partial \epsilon_1\partial \epsilon_2} + \bar{H}_{\epsilon_1\epsilon_2}\right]_{\epsilon_1=\epsilon_2=\epsilon_3=0} \\
& + \epsilon_1\epsilon_3\left[\bar{H}_{\bar{x}'}\frac{\partial^2 \bar{x}'}{\partial \epsilon_1\partial \epsilon_3} + \bar{H}_{\bar{y}'}\frac{\partial^2 \bar{y}'}{\partial \epsilon_1\partial \epsilon_3} + \bar{H}_{\bar{z}'}\frac{\partial^2 \bar{z}'}{\partial \epsilon_1\partial \epsilon_3} + \bar{H}_{\epsilon_1\epsilon_3}\right]_{\epsilon_1=\epsilon_2=\epsilon_3=0} \\
& + \epsilon_1\epsilon_2\epsilon_3\left[\bar{H}_{\bar{x}'}\frac{\partial^3 \bar{x}'}{\partial \epsilon_1\partial \epsilon_2\partial \epsilon_3} + \bar{H}_{\bar{y}'}\frac{\partial^3 \bar{y}'}{\partial \epsilon_1\partial \epsilon_2\partial \epsilon_3} + \bar{H}_{\bar{z}'}\frac{\partial^3 \bar{z}'}{\partial \epsilon_1\partial \epsilon_2\partial \epsilon_3} + \bar{H}_{\epsilon_1\epsilon_2\epsilon_3}\right]_{\epsilon_1=\epsilon_2=\epsilon_3=0} \\
& + \cdots = 0 \qquad (3.05)
\end{aligned}
$$

where all quantities such as $\bar{H}_{\bar{x}'}$, $\bar{H}_{\bar{y}}$, etc. are to be evaluated for $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0$, i.e., on the surface $H_{000}(x,y,z,t) = 0$ (see (3.8)) into which our hull collapses when $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0$. With regard to (3.4) and (2.28) we obtain:

$$
\begin{aligned}
H(x,y,z,t;\epsilon_1,\epsilon_2,\epsilon_3) = {} & (x - x_{c000}(t))\sin\alpha_{000}(t) + (y - y_{c000}(t))\cos\alpha_{000}(t) \\
& + \epsilon_1 \sum_{j,k=0}^{1} \epsilon_2^j \epsilon_3^k \Big[(x - x_{c000}(t))\{\theta_{31jk}(t) + \alpha_{1jk}(t)\}\cos\alpha_{000}(t) \\
& \quad - (y - y_{c000}(t))\{\theta_{31jk}(t) + \alpha_{1jk}(t)\}\sin\alpha_{000}(t) \\
& \quad - x_{c1jk}(t)\sin\alpha_{000}(t) - y_{c1jk}(t)\cos\alpha_{000}(t) - \theta_{1jk}(t)(z - \bar{z}_c') \\
& \quad \mp \delta_{1jk}\cdot h(\{x - x_{c000}(t)\}\cos\alpha_{000}(t) - \{y - y_{c000}(t)\}\sin\alpha_{000}(t), z)\Big] \\
& + \cdots = 0 \qquad (3.06)
\end{aligned}
$$

with $\delta_{100} = 1$, $\delta_{1jk} = 0$ for $j, k = 0, 1$. In writing down the boundary condition on the hull it is convenient to use the abbreviation:

$$
\begin{aligned}
H(x,y,z,t;\epsilon_1,\epsilon_2,\epsilon_3) = {} & H_{000}(x,y,z,t) + \epsilon_1 H_{100}(x,y,z,t) + \epsilon_1\epsilon_2 H_{110}(x,y,z,t) \\
& + \epsilon_1\epsilon_3 H_{101}(x,y,z,t) + \epsilon_1\epsilon_2\epsilon_3 H_{111}(x,y,z,t) + \cdots = 0
\end{aligned}
$$

(3.07)

where on account of (3.6)

$$H_{000}(x,y,z,t) = (x - x_{c000}(t)) \sin \alpha_{000}(t) + (y - y_{c000}(t)) \cos \alpha_{000}(t) = 0 \tag{3.08}$$

$$\begin{aligned} H_{1jk}(x,y,z,t) = {} & (x - x_{c000}(t)) \{ \Theta_{31jk}(t) + \alpha_{1jk}(t) \} \cos \alpha_{000}(t) \\ & -(y - y_{c000}(t)) \{ \Theta_{31jk}(t) + \alpha_{1jk}(t) \} \sin \alpha_{000}(t) \\ & - x_{c1jk}(t) \sin \alpha_{000}(t) - y_{c1jk}(t) \cos \alpha_{000}(t) - \Theta_{11jk}(t)(z - \bar{z}_c') \\ & \mp \delta_{1jk} \cdot h(\{x - x_{c000}(t)\} \cos \alpha_{000}(t) - \{y - y_{c000}(t)\} \sin \alpha_{000}(t), z) = 0 \end{aligned} \tag{3.09}$$

for $j = 0, 1$; $k = 0, 1$ with $\delta_{100} = 1$; $\delta_{1jk} = 0$ otherwise.
We now substitute the expansions (1.13) and (3.7) in the boundary condition (3.3) and get

$$\begin{aligned} & [\epsilon_1 \phi_{100x} + \epsilon_1\epsilon_2 \phi_{110x} + \epsilon_1\epsilon_3 \phi_{101x} + \epsilon_1\epsilon_2\epsilon_3 \phi_{111x} + \cdots][H_{000x} + \epsilon_1 H_{100x} + \epsilon_1\epsilon_2 H_{110x} + \epsilon_1\epsilon_3 H_{101x} + \epsilon_1\epsilon_2\epsilon_3 H_{111x} + \cdots] \\ & + [\epsilon_1 \phi_{100y} + \epsilon_1\epsilon_2 \phi_{110y} + \epsilon_1\epsilon_3 \phi_{101y} + \epsilon_1\epsilon_2\epsilon_3 \phi_{111y} + \cdots][H_{000y} + \epsilon_1 H_{100y} + \epsilon_1\epsilon_2 H_{110y} + \epsilon_1\epsilon_3 H_{101y} + \epsilon_1\epsilon_2\epsilon_3 H_{111y} + \cdots] \\ & + [\epsilon_1 \phi_{100z} + \epsilon_1\epsilon_2 \phi_{110z} + \epsilon_1\epsilon_3 \phi_{101z} + \epsilon_1\epsilon_2\epsilon_3 \phi_{111z} + \cdots][H_{000z} + \epsilon_1 H_{100z} + \epsilon_1\epsilon_2 H_{110z} + \epsilon_1\epsilon_3 H_{101z} + \epsilon_1\epsilon_2\epsilon_3 H_{111z} + \cdots] \\ & + H_{000t} + \epsilon_1 H_{100t} + \epsilon_1\epsilon_2 H_{110t} + \epsilon_1\epsilon_3 H_{101t} + \epsilon_1\epsilon_2\epsilon_3 H_{111t} + \cdots = 0 \end{aligned} \tag{3.10}$$

The terms of zero order lead to the condition

$$H_{000t} = 0 \tag{3.11}$$

and the terms of order $\epsilon_1\epsilon_2{}^j\epsilon_3{}^k (j = 0,1; k = 0,1)$ to the condition

$$\phi_{1jkx} H_{000x} + \phi_{1jky} H_{000y} + \phi_{1jkz} H_{000z} + H_{1jkt} = 0 \qquad \begin{matrix} j = 0,1 \\ k = 0,1 \end{matrix} . \tag{3.12}$$

The conditions (3.11) and (3.12) are to be satisfied on the surface

$$H_{000}(x,y,z,t) \equiv (x - x_{c000}(t)) \sin \alpha_{000}(t) + (y - y_{c000}(t)) \cos \alpha_{000}(t) = 0 \qquad x,y,z \text{ on } A . \tag{3.13}$$

where A is the region whose points have coordinates x,y,z which follow from the transformation formulas (2.28) with $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0$:

$$
\begin{aligned}
x &= \bar{x}' \cos\alpha_{000}(t) + \bar{y}' \sin\alpha_{000}(t) + x_{c000}(t) \\
y &= -\bar{x}' \sin\alpha_{000}(t) + \bar{y}' \cos\alpha_{000}(t) + y_{c000}(t) \\
z &= \bar{z}'
\end{aligned}
\qquad (3.14)
$$

from the coordinates $\bar{x}'$, $\bar{y}'$, $\bar{z}'$ of the region $\bar{A}'$, i.e., A is the region, into which the hull surface collapses for $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0$. First we consider the relation (3.11). With $H_{000}(x,y,z,t)$ from (3.13) the relation (3.11) becomes:

$$
\begin{aligned}
&\dot{\alpha}_{000}(t)\left\{(x - x_{c000}(t))\cos\alpha_{000}(t) - (y - y_{c000}(t))\sin\alpha_{000}(t)\right\} \\
&\quad - \dot{x}_{c000}(t)\cdot\sin\alpha_{000}(t) - \dot{y}_{c000}(t)\cos\alpha_{000}(t) = 0 \qquad x,y,z \text{ on } A.
\end{aligned}
\qquad (3.15)
$$

On account of (2.36) and the equation

$$
\bar{x}'\Big|_{\epsilon_1=\epsilon_2=\epsilon_3=0} = (x - x_{c000}(t))\cos\alpha_{000}(t) + (y - y_{c000}(t))\sin\alpha_{000}(t) \qquad x,y,z \text{ on } A,
$$

which follows from (2.28), relation (3.15) can be written in the form

$$
\dot{\alpha}_{000}(t)\cdot\bar{x}' = 0 \qquad \bar{x}',\bar{y}',\bar{z}' \text{ on } \bar{A}'. \qquad (3.16)
$$

Therefore

$$
\dot{\alpha}_{000}(t) = 0 \quad . \qquad (3.17)
$$

As we have seen in section 2, the relations

$$
\alpha_{000}(t) = 0, \quad x_{c000}(t) = \int_{t_0}^{t} s_{000}(t)\,dt, \quad y_{c000}(t) = 0 \qquad (3.18)
$$

are a consequence of (3.17). With regard to these relations and (3.8), (3.9) the conditions (3.12) yield the boundary conditions:

$$
\begin{aligned}
\phi_{1jky} = &-[\dot{\Theta}_{31jk}(t) + \dot{\alpha}_{1jk}(t)][x - x_{c000}(t)] + [\Theta_{31jk}(t) + \alpha_{1jk}(t)]\,\dot{x}_{c000}(t) \\
&+ \dot{y}_{c1jk}(t) + \dot{\Theta}_{11jk}(t)\cdot(z - \bar{z}'_c) \\
&\mp \delta_{1jk}\cdot h_{\bar{x}'}(x - x_{c000}(t), z)\cdot\dot{x}_{c000}(t)
\end{aligned}
\qquad
\begin{aligned}
&x,y,z \text{ on } A_\pm \\
&j = 0,1 \quad \delta_{100} = 1 \\
&k = 0,1 \quad \delta_{1jk} = 0 \text{ otherwise}
\end{aligned}
\qquad (3.19)
$$

Here the region A for x,y,z is obtained from the region $\bar{A}'$ for $\bar{x}'$, $\bar{y}'$, $\bar{z}'$ with the help of the transformation formulas

$$
\begin{aligned}
x &= \bar{x}' + x_{c000}(t) \\
y &= \bar{y}' \\
z &= \bar{z}'
\end{aligned}
\tag{3.20}
$$

which follow from (3.14) because of (3.18). With regard to (2.37), (3.18) and $\dot{x}_{c000}(t) = s_{000}(t)$, $\dot{\alpha}_{1jk}(t) = \omega_{1jk}(t)$ $j = 0,1$, $k = 0,1$ the boundary conditions on the ship's hull can finally be written in the form:

$$
\begin{aligned}
\phi_{1jky} = {} & s_{000}(t)\Big[\Theta_{31jk}(t) \mp \delta_{1jk} \cdot h_{\bar{x}'}\Big(x - \int_{t_0}^{t} s_{000}(t)\,dt, z\Big)\Big] \\
& - \Big(x - \int_{t_0}^{t} s_{000}(t)\,dt\Big)\big[\omega_{1jk}(t) + \dot{\Theta}_{31jk}(t)\big] \\
& + (z - \bar{z}_c')\,\dot{\Theta}_{11jk}(t) \qquad x,y,z \text{ on } A_\pm \\
& \qquad j = 0,1 \quad \delta_{100} = 1 \\
& \qquad k = 0,1 \quad \delta_{1jk} = 0 \text{ otherwise.}
\end{aligned}
\tag{3.21}
$$

For our purposes it is more convenient to work with the hydrodynamic pressures

$$
p_{1jk}(x,y,z,t) = -\phi_{1jkt}(x,y,z,t) \qquad j = 0,1 \quad k = 0,1 .
$$

If we take in consideration the result of the last section

$$
s_{000}(t) = s_{000} = \text{const.}
$$

we get as the boundary condition on the ship's hull:

$$
\begin{aligned}
p_{1jky} = {} & \mp \delta_{1jk} \cdot s_{000}^2 \cdot h_{\bar{x}'\bar{x}'}\big(x - s_{000}(t-t_0), z\big) + s_{000}\,\dot{\omega}_{1jk}(t) \\
& - 2 s_{000} \cdot \dot{\Theta}_{31jk}(t) + \big[x - s_{000}(t-t_0)\big]\big[\ddot{\Theta}_{31jk}(t) + \dot{\omega}_{1jk}(t)\big] \\
& + (z - \bar{z}_c')\,\ddot{\Theta}_{11jk}(t) \qquad x,y,z \text{ on } A_\pm \\
& \qquad j = 0,1 \quad \delta_{100} = 1 \\
& \qquad k = 0,1 \quad \delta_{1jk} = 0 \text{ otherwise}
\end{aligned}
\tag{3.22}
$$

where the region A for x,y,z follows from the region $\bar{A}'$ for $\bar{x}',\bar{y}',\bar{z}'$ by the transformation formulas (3.20) with $x_{c000}(t) = s_{000}(t-t_0)$.

4. The Boundary Value Problems for the Hydrodynamic Pressure

Now we are able to formulate the motion of our ship as a linear boundary value problem for the hydrodynamic pressures $P_{1jk}(x,y,z,t)$ $j = 0,1,\ k = 0,1$. The differential equations and boundary conditions derived in section 1 and 2 for a theory of first order in ϵ_1 can directly be taken over. Therefore with repsect to the fixed x,y,z-coordinate system we have respectively the following boundary value problems, if we assume that for $t = 0$ the x-coordinate of the center of gravity of the ship is zero. For the dynamic pressure $p_{100}(x,y,z,t)$:

$$\nabla^2 p_{100} = p_{100xx} + p_{100yy} + p_{100zz} = 0 \qquad \begin{cases} -h < z < 0 \\ -B_2 < y < B_1 \\ \text{out of } A \end{cases} \tag{4.01}$$

$$p_{100z}(x,y,z,t) = 0 \qquad \text{on } z = -h \tag{4.02}$$

$$p_{100y}(x,y,z,t) = 0 \qquad \text{on } y = \pm B_i \quad i=1,2 \tag{4.03}$$

$$g\,p_{100z}(x,y,z,t) + p_{100tt}(x,y,z,t) = 0 \qquad \text{on } z = 0 \tag{4.04}$$

$$\begin{aligned} p_{100y} = {} & \mp s_{000}^2\, h_{\bar{x}'\bar{x}'}(x - s_{000}t, z) + s_{000}\,\omega_{100}(t) \\ & - 2 s_{000}\dot{\Theta}_{3100}(t) + [x - s_{000}t][\ddot{\Theta}_{3100}(t) + \dot{\omega}_{100}(t)] \\ & + (z - \bar{z}_c')\,\ddot{\Theta}_{1100}(t) \qquad x,y,z \text{ on } A_{\pm}\,; \end{aligned} \tag{4.05}$$

for the dynamic pressure $P_{110}(x,y,z,t)$:

$$\nabla^2 p_{110} = p_{110xx} + p_{110yy} + p_{110zz} = 0 \qquad \begin{cases} -h < z < 0 \\ -B_2 < y < B_1 \\ \text{out of } A \end{cases} \tag{4.06}$$

$$\begin{aligned} p_{110z}(x,y,z,t) = {} & p_{100x}(x,y,z,t)\, g_x(x,y) \\ & + p_{100y}(x,y,z,t)\, g_y(x,y) \\ & - p_{100zz}(x,y,z,t)\, g(x,y) \qquad \text{on } z = -h \end{aligned} \tag{4.07}$$

$$p_{110y}(x,y,z,t) = 0 \qquad \text{on } y = \pm B_i \quad i=1,2 \tag{4.08}$$

$$g\, p_{110z}(x,y,z,t) + p_{110tt}(x,y,z,t) = 0 \qquad \text{on } z=0 \tag{4.09}$$

$$\begin{aligned} p_{110y} = {} & s_{000}\,\omega_{110}(t) - 2 s_{000}\,\dot{\Theta}_{31110}(t) + [x - s_{000}t][\ddot{\Theta}_{3110}(t) + \dot{\omega}_{110}(t)] \\ & + (z - \bar{z}_c')\,\ddot{\Theta}_{1110}(t) \qquad x,y,z \text{ on } A_{\pm}\ ; \end{aligned} \tag{4.10}$$

for the hydrodynamic pressure $P_{101}(x,y,z,t)$:

$$\nabla^2 p_{101} = p_{101xx} + p_{101yy} + p_{101zz} = 0 \qquad \begin{cases} -h < z < 0 \\ -B_2 < y < B_1 \\ \text{out of } A \end{cases} \tag{4.11}$$

$$p_{101z}(x,y,z,t) = 0 \qquad \text{on } z=-h \tag{4.12}$$

$$\begin{aligned} p_{101y}(x,y,z,t) = {} & \mp p_{100x}(x,y,z,t)\cdot b_{ix}(x,z) \\ & \mp p_{100z}(x,y,z,t)\cdot b_{iz}(x,z) \\ & \pm p_{100yy}(x,y,z,t)\cdot b_i(x,z) \qquad \text{on } y=\pm B_i \quad i=1,2 \end{aligned} \tag{4.13}$$

$$g\, p_{101z}(x,y,z,t) + p_{101tt}(x,y,z,t) = 0 \qquad \text{on } z=0 \tag{4.14}$$

$$\begin{aligned} p_{101y} = {} & s_{000}\,\omega_{101}(t) - 2 s_{000}\,\dot{\Theta}_{3101}(t) + [x - s_{000}t][\ddot{\Theta}_{3101}(t) + \dot{\omega}_{101}(t)] \\ & + (z - \bar{z}_c')\,\ddot{\Theta}_{1101}(t) \qquad x,y,z \text{ on } A_{\pm}\ ; \end{aligned} \tag{4.15}$$

and last for the hydrodynamic pressure $P_{111}(x,y,z,t)$:

$$\nabla^2 p_{111} = p_{111\,xx} + p_{111\,yy} + p_{111\,zz} = 0 \qquad \begin{cases} -h < z < 0 \\ -B_2 < y < B_1 \\ \text{out of } \dot{A} \end{cases} \tag{4.16}$$

$$\begin{aligned} p_{111\,z}(x,y,z,t) &= p_{101\,x}(x,y,z,t)\cdot g_x(x,y) + p_{101\,y}(x,y,z,t)\, g_y(x,y) \\ &\quad - p_{101\,zz}(x,y,z,t)\cdot g(x,y) \qquad \text{on } z=-h \end{aligned} \tag{4.17}$$

$$\begin{aligned} p_{111\,y}(x,y,z,t) &= \mp p_{110\,x}(x,y,z,t)\cdot b_{ix}(x,z) \mp p_{110\,z}(x,y,z,t)\, b_{iz}(x,z) \\ &\quad \pm p_{110\,y}(x,y,z,t)\cdot b_i(x,z) \qquad \text{on } y=\pm B_i \quad i=1,2 \end{aligned} \tag{4.18}$$

$$g\, p_{111\,z}(x,y,z,t) + p_{111\,tt}(x,y,z,t) = 0 \qquad \text{on } z=0 \tag{4.19}$$

$$\begin{aligned} p_{111\,y} &= s_{000}\cdot\omega_{111}(t) - 2s_{000}\,\dot{\Theta}_{3111}(t) + [x - s_{000}t][\ddot{\Theta}_{3111}(t) + \dot{\omega}_{111}(t)] \\ &\quad + (z - \bar{z}_c')\,\ddot{\Theta}_{1111}(t) \qquad x,y,z \text{ on } A_{\pm}\,. \end{aligned} \tag{4.20}$$

Here the region A for x,y,z follows from the region $\bar{A}'$ for $\bar{x}',\bar{y}',\bar{z}'$ by the transformation formulas

$$\begin{aligned} x &= \bar{x}' + s_{000}t \\ y &= \bar{y}' \\ z &= \bar{z}' \end{aligned}$$

A boundary value problem which is also interesting in connection with our problem is that, in which the fluid extends horizontally to infinity ($B_1 = B_2 = \infty$) but the fluid depth is variable. Here we have for the dynamic pressure $P_{11}(x,y,z,t)$ which describes the influence of the variable depth of the water in analogy to (4.6) - (4.10) the boundary value problem:

$$\nabla^2 p_{11} = p_{11\,xx} + p_{11\,yy} + p_{11\,zz} = 0 \qquad \begin{cases} -h < z < 0 \\ \text{out of } A \end{cases} \tag{4.21}$$

$$\begin{aligned} p_{11\,z}(x,y,z,t) &= p_{10\,x}(x,y,z,t)\cdot g_x(x,y) + p_{10\,y}(x,y,z,t)\, g_y(x,y) \\ &\quad - p_{10\,zz}(x,y,z,t)\, g(x,y) \qquad \text{on } z=-h \end{aligned} \tag{4.22}$$

$$g\, p_{11z}(x,y,z,t) + p_{11tt}(x,y,z,t) = 0 \qquad \text{on } z = 0 \qquad (4.23)$$

$$p_{11y} = s_{00}\cdot\omega_{11}(t) - 2s_{00}\dot{\Theta}_{311}(t) + [x - s_{00}t][\ddot{\Theta}_{311}(t) + \dot{\omega}_{11}(t)] + (z - \bar{z}_c')\ddot{\Theta}_{111}(t) \qquad x,y,z \text{ on } A_{\pm} \qquad (4.24)$$

Here we have a two parameter perturbation theory and therefore all the physical variables have only two indices. The dynamic pressure $P_{10}(x,y,z,t)$ whose derivatives appear in (4.22) is identical with the solution $P_{100}(x,y,z,t)$ of the boundary value problem (4.1) - (4.2) and (4.3) - (4.5) except that the third index of all the physical variables is omitted.

The formulation of the boundary problem for a canal of variable breadth but infinite depth is directly evident from (4.11) - (4.15).

5. The Solution of Our Boundary Value Problems

Now we are able to solve the various boundary value problems. For this purpose we introduce the following Green's functions: The function $G(x,y,z,t;\xi,\eta,\zeta;\Omega)$ satisfies the following conditions:

$$\nabla^2 G = G_{xx} + G_{yy} + G_{zz} = -4\pi \exp[-i\Omega t]\delta(x-s_{00}t-\xi)\delta(y-\eta)\delta(z-\zeta) \qquad \begin{array}{l} -h < z < 0 \\ -b_2 < y < b_1 \end{array} \qquad (5.01)$$

$$G_z = 0 \qquad \text{on } z = -h \qquad (5.02)$$

$$g G_z + G_{tt} = 0 \qquad \text{on } z = 0 \qquad (5.03)$$

$$G_y = 0 \qquad \text{on } y = \pm b_i \quad i = 1,2 \qquad (5.04)$$

together with some radiation conditions. If we introduce $G(x,y,z;\xi,\eta,\zeta;\Omega)$ by the equation

$$G(x,y,z,t;\xi,\eta,\zeta;\Omega) = \exp[-i\Omega t]\cdot\bar{G}(x - s_{00}t,y,z;\xi,\eta,\zeta;\Omega)$$

the boundary value problem (5.1) - (5.5) transforms into

$$\nabla^2\bar{G} = \bar{G}_{\bar{x}\bar{x}} + \bar{G}_{\bar{y}\bar{y}} + \bar{G}_{\bar{z}\bar{z}} = -4\pi\,\delta(\bar{x}-\xi)\delta(\bar{y}-\eta)\delta(\bar{z}-\zeta) \qquad \begin{matrix} -h<\bar{z}<0 \\ -B_2<\bar{y}<B_1 \end{matrix} \tag{5.01)'}$$

$$\bar{G}_{\bar{z}} = 0 \qquad \text{on } \bar{z}=-h \tag{5.02)'}$$

$$g\bar{G}_{\bar{z}} + (s_{\dots})^2\bar{G}_{\bar{x}\bar{x}} + 2i\Omega s_{\dots}\bar{G}_{\bar{x}} - \Omega^2\bar{G} = 0 \qquad \text{on } \bar{z}=0 \tag{5.03)'}$$

$$\bar{G}_{\bar{y}} = 0 \qquad \text{on } \bar{y}=\pm B_i;\ i=1,2 \tag{5.04)'}$$

together with some radiation conditions.

The Green's function which satisfies (5.1) - (5.3) respectively can be found in reference 4. For later reference we designate this function by $G^{\infty}(x,y,z,t;\xi,\eta,\zeta;\Omega)$ resp. $\bar{G}^{\infty}(x,y,z;\xi,\eta,\zeta;\Omega)$. With the help of the method of images described in Lunde(7), the Green's function which satisfies (5.1) - (5.4) resp. (5.1)'- (5.4)' is obtainable.

One other Green's function we need is the function $G^{(0)}(x,y,z,t;\xi,\eta;\Omega)$ which satisfies the conditions

$$\nabla^2 G^{(0)} = G^{(0)}_{xx} + G^{(0)}_{yy} + G^{(0)}_{zz} = 0 \qquad \begin{matrix} -h<z<0 \\ -B_2<y<B_1 \end{matrix} \tag{5.05}$$

$$G^{(0)}_z = \exp[-i\Omega t]\,\delta(x-s_{\dots}t-\xi)\,\delta(y-\eta) \qquad \text{on } z=-h \tag{5.06}$$

$$gG^{(0)}_z + G^{(0)}_{tt} = 0 \qquad \text{on } z=0 \tag{5.07}$$

$$G^{(0)}_y = 0 \qquad \text{on } y=\pm B_i;\ i=1,2 \tag{5.08}$$

together with some radiation conditions. If we introduce $\bar{G}^{(0)}(\bar{x},\bar{y},\bar{z};\xi,\eta;\Omega)$ by the equation

$$G^{(0)}(x,y,z,t;\xi,\eta;\Omega) = \exp[-i\Omega t]\,\bar{G}^{(0)}(x-s_{\dots}t,y,z;\xi,\eta;\Omega)$$

the boundary value problem (5.5) to (5.8) transforms into

$$\nabla^2 \bar{G}^{(0)} = \bar{G}^{(0)}_{\bar{x}\bar{x}} + \bar{G}^{(0)}_{\bar{y}\bar{y}} + \bar{G}^{(0)}_{\bar{z}\bar{z}} = 0 \qquad \begin{array}{l} -h < \bar{z} < 0 \\ -B_2 < \bar{y} < B_1 \end{array} \qquad (5.05)'$$

$$\bar{G}^{(0)}_{\bar{z}} = \delta(\bar{x} - \xi)\,\delta(\bar{y} - \eta) \qquad \text{on } \bar{z} = -h \qquad (5.06)'$$

$$g\bar{G}^{(0)}_{\bar{z}} + (s_{000})^2 \bar{G}^{(0)}_{\bar{x}\bar{x}} + 2i\Omega s_{000} \bar{G}^{(0)}_{\bar{x}} - \Omega^2 \bar{G}^{(0)} = 0 \qquad \text{on } \bar{z} = 0 \quad (5.07)'$$

$$\bar{G}^{(0)}_{\bar{y}} = 0 \qquad \text{on } \bar{y} = \pm B_i \quad i = 1,2 \quad (5.08)'$$

together with some radiation conditions.

If we first consider the Green's function $G^{\infty(0)}(x,y,z,t;\xi,\eta;\Omega)$ resp. $\bar{G}^{\infty(0)}(\bar{x},\bar{y},\bar{z};\xi,\eta;\Omega)$ which satisfies the conditions (5.5) - (5.7) resp. (5.5)'- (5.7)', this function is found in reference 4 and can be determined by a method similar to that, by which the function $G^{\infty}(x,y,z,t;\xi,\eta,\zeta;\Omega)$ resp. $\bar{G}^{\infty}(\bar{x},\bar{y},\bar{z};\xi,\eta,\zeta;\Omega)$ is obtained. Then $G^{(0)}(x,y,z,t;\xi,\eta;\Omega)$ resp. $\bar{G}^{(0)}(\bar{x},\bar{y},\bar{z};\xi,\eta,;\Omega)$ which also satisfies the condition (5.8) resp. (5.8)' can be determined by applying the method of images.

For future reference we denote by

$$G^{*(0)}(x,y,z,t;\xi,\eta;\Omega) \quad \text{resp.} \quad \bar{G}^{*(0)}(\bar{x},\bar{y},\bar{z};\xi,\eta;\Omega)$$

the Green's functions, which satisfy the above conditions with specially $S_{000} = 0$ in (5.6) resp. (5.7)'. Analogous for the functions

$$G^{*\infty(0)}(x,y,z,t;\xi,\eta;\Omega) \quad \text{resp.} \quad \bar{G}^{*\infty(0)}(\bar{x},\bar{y},\bar{z};\xi,\eta;\Omega) .$$

The last two Green's functions $G^{(1)}(x,y,z,t;\xi,\zeta,\Omega)$ and $G^{(2)}(x,y,z,t;\xi,\zeta;\Omega)$ needed must satisfy the following conditions:

$$\nabla^2 G^{(i)} = G^{(i)}_{xx} + G^{(i)}_{yy} + G^{(i)}_{zz} = 0 \qquad i=1,2 \qquad \begin{array}{l} -h < z < 0 \\ -B_2 < y < B_1 \end{array} \tag{5.09}$$

$$G^{(i)}_{z} = 0 \qquad \text{on } z=-h \tag{5.10}$$

$$g\,G^{(i)}_{z} + G^{(i)}_{tt} = 0 \qquad i=1,2 \qquad \text{on } z=0 \tag{5.11}$$

and

$$G^{(1)}_{y} = \exp[-i\Omega t]\,\delta(x-s_{000}t-\xi)\,\delta(z-\zeta) \qquad \text{on } y=B_1 \tag{5.12}$$

$$G^{(1)}_{y} = 0 \qquad \text{on } y=-B_2 \tag{5.13}$$

respectively

$$G^{(2)}_{y} = 0 \qquad \text{on } y=B_1 \tag{5.14}$$

$$G^{(2)}_{y} = \exp[-i\Omega t]\,\delta(x-s_{000}t-\xi)\,\delta(z-\zeta) \qquad \text{on } y=-B_2\,. \tag{5.15}$$

together with some radiation conditions. Again introducing $\bar{G}^{(i)}(\bar{x},\bar{y},\bar{z},\xi,\zeta;\Omega)$ by

$$G^{(i)}(x,y,z,t;\xi,\zeta;\Omega) = \exp[-i\Omega t]\,\bar{G}^{(i)}(x-s_{000}t,y,z;\xi,\zeta;\Omega)$$

the boundary value problem (5.9) - (5.15) transforms in:

$$\nabla^2 \bar{G}^{(i)} = \bar{G}^{(i)}_{\bar{x}\bar{x}} + \bar{G}^{(i)}_{\bar{y}\bar{y}} + \bar{G}^{(i)}_{\bar{z}\bar{z}} = 0 \qquad \begin{array}{l} -h < \bar{z} < 0 \\ -B_2 < \bar{y} < B_1 \end{array} \tag{5.09'}$$

$$\bar{G}^{(i)}_{\bar{z}} = 0 \qquad i=1,2 \qquad \text{on } \bar{z}=-h \tag{5.10'}$$

$$g\bar{G}^{(i)}_{\bar{z}} - (s_{000})^2\bar{G}^{(i)}_{\bar{x}\bar{x}} + 2i\Omega s_{000}\bar{G}^{(i)}_{\bar{x}} - \Omega^2\bar{G}^{(i)} = 0 \qquad \text{on } \bar{z}=0 \tag{5.11'}$$

and

$$\bar{G}^{(1)}_{\bar{y}} = \delta(\bar{x}-\xi)\,\delta(\bar{z}-\zeta) \qquad \text{on } \bar{y} = \mathcal{B}_1 \qquad (5.12)'$$

$$G^{(1)}_{\bar{y}} = 0 \qquad \text{on } \bar{y} = -\mathcal{B}_2 \qquad (5.13)'$$

respectively

$$\bar{G}^{(2)}_{\bar{y}} = 0 \qquad \text{on } \bar{y} = \mathcal{B}_1 \qquad (5.14)'$$

$$\bar{G}^{(2)}_{\bar{y}} = \delta(\bar{x}-\xi)\,\delta(\bar{z}-\zeta) \qquad \text{on } \bar{y} = -\mathcal{B}_2 \qquad (5.15)'$$

together with some radiation conditions. The Green's functions $G^{\infty(i)}(x,y,z,t;\xi,\zeta;\Omega)$ [resp. $\bar{G}^{\infty(i)}(\bar{x},\bar{y},\bar{z};\xi,\zeta;\Omega)$] satisfying the conditions (5.9) - (5.11) [resp. (5.9)' - (5.11)'] and (5.12) [resp. (5.12)'] respectively (5.15) [resp. (5.15)'] can be determined by a method similar that by which the function $G^{\infty}(x,y,z,t;\xi,\eta,\zeta;\Omega)$ [resp. $\bar{G}^{\infty}(\bar{x},\bar{y},\bar{z};\xi,\eta,\zeta;\Omega)$] is obtained. These functions we denote by $G^{\infty(i)}(x,y,z,t;\xi,\zeta;\Omega)$ resp. $\bar{G}^{\infty(i)}(\bar{x},\bar{y},\bar{z};\xi,\zeta;\Omega)$ $i = 1, 2$. Then the functions $G^{(i)}(x,y,z,t;\xi,\zeta;\Omega)$ [resp. $\bar{G}^{(i)}(\bar{x},\bar{y},\bar{z};\xi,\zeta;\Omega)$] satisfying also the conditions (5.13) [resp. (5.13)'] respectively (5.14) [resp. (5.14)'] can again be found by applying the method of images.

Further we denote by $G^{*(i)}(x,y,z,t;\xi,\zeta;\Omega)$ resp. $G^{*(i)}(\bar{x},\bar{y},\bar{z};\xi,\zeta;\Omega)$ $i = 1,2$ the Green's functions which satisfy the above conditions with specially $s_{000} = 0$ in (5.12) [resp. (5.11)'] resp. (5.15) [resp. (5.11)']. Analogous for the functions $G^{*\infty(i)}(x,y,z,t;\xi,\zeta;\Omega)$ resp. $\bar{G}^{*\infty(i)}(\bar{x},\bar{y},\bar{z};\xi,\zeta;\Omega)$ $i = 1,2$. These functions are necessary when the motion of a ship along a wall is considered.

The methods for obtaining the Green's functions will not be given in detail here, only these explanations be enough.

5.1 The Solution of the Boundary Value Problem (5.1) - (5.5)

From the conditions (5.1) - (5.5) and the differential equations for the physical terms $\omega_{100}(t)$, $\Theta_{3100}(t)$, $\Theta_{1100}(t)$, derived in section 7, together with the initial conditions at the time $t = -\infty$, we see that

$$\omega_{100}(t) = 0 \;, \quad \Theta_{3100}(t) = \text{const.} \;, \quad \Theta_{4100}(t) = \text{const.} \;, \qquad (5.16)$$

and with (5.16) the boundary value problem (4.1) - (4.5) reduces exactly to the problem of finding the hydrodynamic pressure for a ship, moving horizontally with the velocity parallel to the vertical and parallel walls of a canal of constant breadth $B_1 + B_2$ and constant depth h. The solution of this problem is known and for instance found in the paper of Kostjukow[6] except that Kostjukow refers to the moving $\bar{x}$, $\bar{y}$, $\bar{z}$-coordinate system. If there are no canal walls ($B_1 = B_2 = \infty$) the solution $\phi_{10}(x,y,z,t)$ of this problem can be found in Lunde[7] as well as in many other papers on shallow water ship waves. We have only to transform the formulas of the other authors to the fixed x,y,z-coordinate system.

5.2 The Solution of the Boundary Value Problem (4.6) - (4.16)

The difficulties which arise for the solution of the other boundary value problems lie specially in the fact, that in general there is a dependency on time in some of our boundary conditions which can not be omitted by referring to the moving $\bar{x}$,$\bar{y}$,$\bar{z}$-coordinate system, a method which succeeded in solving the boundary value problem (4.1) - (4.6). Therefore, in these non-steady state cases, we have to use another method. Now the boundary conditions (4.7) and (4.10) suggest the use of potentials which are produced by surface distributions on the surfaces A and z = -h. But according to (4.7) and (4.10) the surface density must depend on the time t and now the free surface condition (4.9) introduces some difficulties. To avoid these difficulties we will first assume surface densities whose time dependence is harmonic with frequency Ω and then generalize by multiplying with a frequency spectrum and integrating with respect to Ω from $\Omega = -\infty$ to $\Omega = +\infty$. Therefore in the non-steady state case where g(x,y) is really dependent on x and $\lim_{x \to \pm\infty} g(x,y) = 0$ as a necessary condition we assume the function $P_{110}(x,y,z,t)$ to have the following form:

$$p_{110}(x,y,z,t) = \int_{-\infty}^{\infty} \exp[-i\Omega t]\Big[\iint_{A'} \mu_{110}(\xi,\zeta;\Omega)\,\bar{G}_{\eta}(x-s_{000}t,y,z;\xi,0,\zeta;\Omega)\,d\xi\,d\zeta$$
$$+ \int_{-\infty}^{\infty}\int_{-B_2}^{B_1} \sigma^{(0)}_{110}(\xi,\eta;\Omega)\,\bar{G}^{*(0)}(x,y,z;\xi,\eta;\Omega)\,d\xi\,d\eta\Big]\,d\Omega \qquad (5.17)$$

where the functions $\mu_{110}(\xi,\zeta;\Omega)$ and $\sigma^{(0)}_{110}(\xi,\eta;\Omega)$ have to be determined such, that the conditions (4.7) and (4.10) are satisfied.

If $g(x,y)$ is independent of x, $g(x,y) \equiv g(y)$, our boundary value problem becomes independent of time t when referred to the moving $\bar{x},\bar{y},\bar{z}$-coordinate system. With respect to this system we assume the hydrodynamic pressure $\bar{P}_{110}(\bar{x},\bar{y},\bar{z})$ to have the form:

$$\bar{p}_{110}(\bar{x},\bar{y},\bar{z}) = \iint_{\bar{A}'} \mu_{110}(\xi,\zeta)\,\bar{G}_{\eta}(\bar{x},\bar{y},\bar{z};\xi,0,\zeta;0)\,d\xi\,d\zeta$$

$$+ \int_{-\infty}^{\infty}\int_{-\delta_2}^{\delta_1} \sigma^{(0)}_{110}\,\bar{G}^{(0)}(\bar{x},\bar{y},\bar{z};\xi,\eta;0)\,d\xi\,d\eta \,. \qquad (5.17)'$$

Again the functions $\mu_{110}(\xi,\zeta)$ and $\sigma^{(0)}_{110}(\xi,\eta)$ have to be determined so that the conditions (4.7) and (4.10) are satisfied. In the next section we will derive integral equations for these functions.

If generally $g(x,y) = g_o(y) + g_1(x,y)$ with $\lim\limits_{x\to\pm\infty}(x,y) = 0$ as a necessary condition the solution can be obtained by superposition of (5.17) and (5.17)'.

5.3 The Solution of the Boundary Value Problem (4.11) - (4.15)

If one of the two functions $b_1(x,z)$ and $b_2(x,z)$ is dependent on x, and $\lim\limits_{x\to+\infty} b_1(x,z) = \lim\limits_{x\to+\infty} b_2(x,z) = 0$ as a necessary condition the problem is non-steady state and the same considerations as before lead here to the following assumption on the form of function $P_{101}(x,y,zt)$:

$$p_{101}(x,y,z,t) = \int_{-\infty}^{\infty} \exp[-i\Omega t]\Big[\iint_{\bar{A}'} \mu_{101}(\xi,\zeta;\Omega)\,\bar{G}_{\eta}(x-s_{\infty}t,y,z;\xi,0,\zeta;\Omega)\,d\xi\,d\zeta$$

$$+\int_{-\infty}^{\infty}\int_{-h}^{0}\Big\{\sigma^{(1)}_{101}(\xi,\zeta;\Omega)\,\bar{G}^{*(1)}(x,y,z;\xi,\zeta;\Omega)$$

$$+\,\sigma^{(2)}_{101}(\xi,\zeta;\Omega)\,\bar{G}^{*(2)}(x,y,z;\xi,\zeta;\Omega)\Big\}\,d\xi\,d\zeta\Big]\,d\Omega \,. \qquad (5.18)$$

The functions $\mu_{101}(\xi,\zeta,\Omega)$, $\sigma^{(1)}_{101}(\xi,\zeta;\Omega)$ and $\sigma^{(2)}_{101}(\xi,\zeta;\Omega)$ have to be determined such that the boundary conditions (4.13) and (4.15) are satisfied.

If the functions $b_1(x,z)$ and $b_2(x,z)$ are independent of x, $b_1(x,z) \equiv b_1(z)$ and $b_2(x,z) \equiv b_2(z)$, our boundary value problem becomes independent of time t when referred to the moving $\bar{x}$, $\bar{y}$, $\bar{z}$-coordinate system. Therefore with respect to this system we assume the following form for the hydrodynamic pressure $\bar{P}_{101}(\bar{x},\bar{y},\bar{z})$:

$$\bar{P}_{101}(\bar{x},\bar{y},\bar{z}) = \iint_{\bar{A}'} \mu_{101}(\xi,\zeta)\,\bar{G}_{\eta}(\bar{x},\bar{y},\bar{z};\xi,0,\zeta;0)\,d\xi\,d\zeta$$
$$+\int_{-\infty}^{\infty}\int_{-h}^{0}\Big\{\sigma_{101}^{(1)}(\xi,\zeta)\,\bar{G}^{(1)}(\bar{x},\bar{y},\bar{z};\xi,\zeta;0)$$
$$+\sigma_{101}^{(2)}(\xi,\zeta)\,\bar{G}^{(2)}(\bar{x},\bar{y},\bar{z};\xi,\zeta;0)\Big\}\,d\xi\,d\zeta \qquad (5.18)'$$

Here also the functions $\mu_{101}(\xi,\zeta)$, $\sigma_{101}^{(1)}(\xi,\zeta)$ and $\sigma_{101}^{(2)}(\xi,\zeta)$ have to be determined such that the boundary conditions (4.13) and (4.15) are satisfied.

If generally $b_1(x,z) = b_{10}(z) + b_{11}(x,z)$ and $b_2(x,z) = b_{20}(z) + b_{21}(x,z)$ with $\lim\limits_{x\to+\infty} b_{11}(x,z) = \lim\limits_{x\to+\infty} b_{21}(x,z) = 0$ as necessary condition, the solution is obtained by superposition of (5.18) and (5.18)'.

5.4 The Solution of the Boundary Value Problem (4.16) - (4.20)

This is the most complicated problem in connection with the considered motion of the ship in a canal of variable breadth and depth. But its solution follows the same line as before. In the non-steady state case, where at least one of the three functions $g(x,y)$, $b_1(x,z)$ and $b_2(x,z)$ is dependent on x and

$$\lim_{x\to\pm\infty} g(x,y) = 0, \quad \lim_{x\to\pm\infty} b_1(x,z) = \lim_{x\to\pm\infty} b_2(x,z) = 0 \text{ as a}$$

necessary condition, we assume the solution $P_{111}(x,y,z,t)$ of the boundary value problem (4.16) - (4.20) to be of the form:

$$\mathfrak{p}_{111}(x,y,z,t) = \int_{-\infty}^{\infty} \exp[-i\Omega t]\Big[\iint_{A'} \mu_{111}(\xi,\zeta;\Omega)\bar{G}_{\eta}(x-s_{00}t;y,z;\xi,0,\zeta;\Omega)\,d\xi d\zeta$$

$$+\int_{-\infty}^{\infty}\int_{-\delta_2}^{\delta_1} \sigma_{111}^{(0)}(\xi,\eta;\Omega)\,\bar{G}^{*(0)}(x,y,z;\xi,\eta;\Omega)\,d\xi d\eta$$

$$+\int_{-\infty}^{\infty}\int_{-h}^{0}\Big\{\sigma_{111}^{(1)}(\xi,\zeta;\Omega)\,\bar{G}^{*(1)}(x,y,z;\xi,\zeta;\Omega)$$

$$+\sigma_{111}^{(2)}(\xi,\zeta;\Omega)\bar{G}^{*(2)}(x,y,z;\xi,\zeta;\Omega)\Big\}d\xi d\zeta\Big]d\Omega \quad (5.19)$$

and now the boundary conditions (4.17), (4.18) and (4.20) will result in equations for the determination of the four functions $\mu_{111}(\xi,\zeta;\Omega)$, $\sigma_{111}^{(0)}(\xi,\eta;\Omega)$, $\sigma_{111}^{(1)}(\xi,\zeta;\Omega)$ and $\sigma_{111}^{(2)}(\xi,\zeta;\Omega)$.

If the three functions $g(x,y)$, $b_1(x,z)$ and $b_2(x,z)$ are independent of x, $g(x,y) \equiv g(y)$, $b_1(x,z) \equiv b_1(z)$ and $b_2(x,z) \equiv b_2(z)$, our boundary value problem becomes stationary when referred to the moving $\bar{x},\bar{y},\bar{z}$-coordinate system. Here we assume the hydrodynamic pressure $\bar{P}_{111}(\bar{x},\bar{y},\bar{z})$ to have the following form:

$$\bar{\mathfrak{p}}_{111}(\bar{x},\bar{y},\bar{z}) = \iint_{A'} \mu_{111}(\xi,\zeta)\,\bar{G}_{\eta}(\bar{x},\bar{y},\bar{z};\xi,0,\zeta;0)\,d\xi d\zeta$$

$$+\int_{-\infty}^{\infty}\int_{-\delta_2}^{\delta_1}\sigma_{111}^{(0)}(\xi,\eta)\,\bar{G}^{(0)}(\bar{x},\bar{y},\bar{z};\xi,\eta;0)\,d\xi d\eta$$

$$+\int_{-\infty}^{\infty}\int_{-h}^{0}\{\sigma_{111}^{(1)}(\xi,\zeta)\bar{G}^{(1)}(\bar{x},\bar{y},\bar{z};\xi,\zeta;0)+\sigma_{111}^{(2)}(\xi,\zeta)\bar{G}^{(2)}(\bar{x},\bar{y},\bar{z};\xi,\zeta;0)\}d\xi d\zeta \quad (5.19)'$$

Again the functions $\mu_{111}(\xi,\zeta)$, $\sigma_{111}^{(0)}(\xi,\eta)$, $\sigma_{111}^{(1)}(\xi,\zeta)$ and $\sigma_{111}^{(2)}(\xi,\zeta)$ have to be determined such that the boundary conditions (4.17), (4.18) and (4.20) are satisfied.

If generally $g(x,y) = g_0(y) + g_1(x,y)$, $b_1(x,z) = b_{10}(z) + b_{11}(x,z)$, $b_2(x,z) = b_{20}(z) + b_{21}(x,z)$ with

$$\lim_{x\to\pm\infty} g(x,y) = \lim_{x\to\pm\infty} b_1(x,z) = \lim_{x\to\pm\infty} b_2(x,z) = 0 \quad \text{as a}$$

necessary condition, the solution is obtained by superposition of (5.19) and (5.19)'.

6. The Integral Equations for Our Surface Distributions

Now the distribution functions in the formulas (5.17), (5.18) and (5.19) have to be determined in such a way that the boundary conditions on the canal bottom and the canal walls are fulfilled. Here only the boundary conditions on the ship's hull makes some difficulties, arising from the first term on the right-hand side of (5.17) - (5.19). The boundary conditions on the ship's hull require the differentiation of $P_{1jk}(j,k = 0,1)$ with respect to y and then a limit process such that the point (x,y,z) goes to a point of A. For the first term in $P_{1jk}(j,k = 0,1)$ such a limit process is not possible, since the function $\bar{G}_{\eta y}$ clearly becomes singular at $x = \xi$, $z = \zeta$ and $y = 0$. This difficulty can be avoided by transforming this integral term before permitting the point (x,y,z) to lie on A in a way described in Peters and Stoker[9]. Here we will only give the formula without any details of derivation. We have for (x,y,z) and A:

$$\iint_{\bar{A}'} \mu(\xi,\zeta;\Omega)\, \bar{G}_{\eta y}(x - s_{\circ\circ\circ}t, y, z; \xi, 0, \zeta; \Omega)\, d\xi\, d\zeta$$

$$= -\iint_{\bar{A}'} [\mu_\xi(\xi,\zeta;\Omega)\, \bar{G}_\xi(x - s_{\circ\circ\circ}t, y, z; \xi, 0, \zeta; \Omega)$$

$$+ \mu_\zeta(\xi,\zeta;\Omega)\, \bar{G}_\zeta(x - s_{\circ\circ\circ}t, y, z; \xi, 0, \zeta; \Omega)]\, d\xi\, d\zeta$$

$$+ \int_{\bar{C}'} \mu(\xi,\zeta;\Omega) \frac{\partial}{\partial n} \bar{G}(x - s_{\circ\circ\circ}t, y, z; \xi, 0, \zeta; \Omega)\, ds \qquad (6.01)$$

in which $\bar{C}'$ represents the boundary of $\bar{A}'$ and $\partial\bar{G}/\partial n$ the derivative of $\bar{G}$ in the direction of the outward normal on $\bar{C}'$ with respect to ξ, ζ. On the right side of this formula only first derivatives of $\bar{G}$ occur and here we can therefore permit the point (x,y,z) to approach a point of A, and then the integral over $\bar{A}'$ would exist in the Cauchy sense. We assume this transformation to be carried out in the first terms on the right-hand side in the formulas for P_{1jky} $(j,k = 0,1)$ needed in the boundary conditions on the ship's hull. Then we are able to formulate integral equations for the distribution functions in (5.17) - (5.19).

6.1 The Distribution Functions of P_{110}

We first consider the boundary condition (4.7). With $P_{110}(x,y,z,t)$ from (5.17) this gives

$$\int_{-\infty}^{\infty} \exp[-i\Omega t]\,\sigma^{(0)}_{110}(x,y;\Omega)\,d\Omega = p_{100x}(x,y,-h,t)\,g_x(x,y) + p_{100y}(x,y,-h,t)\,g_y(x,y) - p_{100zz}(x,y,-h,t)\,g(x,y)$$

$$-\infty < x < \infty \qquad -B_2 < y < B_1 \qquad (6.02)$$

and the inversion formula of the Fourier transformation yields:

$$\sigma^{(0)}_{110}(x,y;\Omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp[i\Omega t]\Big[p_{100x}(x,y,-h,t)\,g_x(x,y) + p_{100y}(x,y,-h,t)\,g_y(x,y) + p_{100zz}(x,y,-h,t)\,g(x,y)\Big]\,dt$$

$$-\infty < x < \infty \qquad -B_2 < y < B_1 \qquad (6.03)$$

The function $\sigma^{(0)}_{110}(x,y;\Omega)$ is therefore determined explicitely. With this formula for $\sigma^{(0)}_{110}(x,y;\Omega)$ in mind and (6.1) the boundary condition (4.10) results in the following integral equation for $\mu_{110}(\xi,\zeta;\Omega)$:

$$\int_{-\infty}^{\infty} \exp[-i\Omega t]\Big[-\iint_{\bar{A}'}\{\mu_{110\xi}(\xi,\zeta;\Omega)\,\bar{G}_{\xi}(\bar{x}',0,\bar{z}';\xi,0,\zeta;\Omega)$$

$$+\mu_{110\zeta}(\xi,\zeta;\Omega)\,\bar{G}_{\zeta}(\bar{x}',0,\bar{z}';\xi,0,\zeta;\Omega)\}\,d\xi\,d\zeta$$

$$+\int_{\bar{C}'}\mu_{110}(\xi,\zeta;\Omega)\frac{\partial}{\partial n}\bar{G}(\bar{x}',0,\bar{z}';\xi,0,\zeta;\Omega)\,ds$$

$$+\int_{-\infty}^{\infty}\int_{-\mathfrak{Z}_2}^{\mathfrak{Z}_1}\sigma_{110}^{(0)}(\xi,\eta;\Omega)\,\bar{G}^{*(0)}(\bar{x}'+s_{000}t,0,\bar{z}';\xi,\eta;\Omega)\,d\xi\,d\eta\Big]d\Omega$$

$$= s_{000}\,\omega_{110}(t) - 2s_{000}\,\dot{\Theta}_{3110}(t) + \bar{x}'[\ddot{\Theta}_{3110}(t) + \dot{\omega}_{110}(t)] + (\bar{z}'-\bar{z}_c')\ddot{\Theta}_{1110}(t)$$

$$\bar{x}',\bar{z}' \text{ on } \bar{A}' \,. \qquad (6.04)$$

In the steady state case with the hydrodynamical pressure $\bar{P}_{110}(\bar{x},\bar{y},\bar{z})$ from (5.17)' the boundary condition (4.7) now with respect to the moving coordinate system gives:

$$\sigma_{110}^{(0)}(\bar{x},\bar{y}) = \bar{P}_{100\bar{y}}(\bar{x},\bar{y},-h)\cdot g_{\bar{y}}(\bar{y}) - \bar{P}_{100\bar{z}\bar{z}}(\bar{x},\bar{y},-h)\,g(\bar{y}) \qquad \begin{array}{l}-\infty < \bar{x} < \infty \\ -\mathfrak{Z}_2 < \bar{y} < \mathfrak{Z}_1 \,.\end{array} \qquad (6.05)$$

The function $\sigma_{110}^{(0)}(x,y)$ is therefore determined explicitely. Now the boundary condition (4.10) results in the following integral equation for $\mu_{110}(\xi,\zeta)$:

$$-\iint_{\bar{A}'}\{\mu_{110\xi}(\xi,\zeta)\,\bar{G}_{\xi}(\bar{x}',0,\bar{z}';\xi,0,\zeta;0) + \mu_{110\zeta}(\xi,\zeta)\,\bar{G}_{\zeta}(\bar{x}',0,\bar{z}';\xi,0,\zeta;0)\}\,d\xi\,d\zeta$$

$$+\int_{\bar{C}'}\mu_{110}(\xi,\zeta)\frac{\partial}{\partial n}\bar{G}(\bar{x}',0,\bar{z}';\xi,0,\zeta;0)\,ds$$

$$+\int_{-\infty}^{\infty}\int_{-\mathfrak{Z}_2}^{\mathfrak{Z}_1}\sigma_{110}^{(0)}(\xi,\eta)\,\bar{G}^{(0)}(\bar{x}',0,\bar{z}';\xi,\eta;0)\,d\xi\,d\eta = 0 \qquad \bar{x}',\bar{z}' \text{ on } \bar{A}', \qquad (6.06)$$

where we have used the result of section 7 that under the present conditions

$$\omega_{110}(t) = 0\,, \quad \Theta_{3110}(t) = \text{const.}\,, \quad \Theta_{1110}(t) = \text{const.}$$

Of special interest is the case where the plane $y = 0$ is a symmetry plane of our canal. If $B_1 = B_2 = B$ and $g(x,y)$ an even function of y:

$$g(x,-y) = g(x,y)$$

then we have generally

$$\omega_{110}(t) = \Theta_{3110}(t) = \Theta_{1110}(t) = 0$$

and the right-hand side of the boundary condition (4.10) vanishes. At the same time the functions $P_{110}(x,y,z,t)$ from (5.17) resp. $\bar{P}_{110}(\bar{x},\bar{y},\bar{z})$ from (5.17)' with

$$\mu_{110}(\xi,\zeta;\Omega) \equiv 0 \qquad \text{resp.} \qquad \mu_{110}(\xi,\zeta) \equiv 0$$

become even functions of y resp. $\bar{y}$ and therefore the boundary condition (4.10) is evidently satisfied by the hydrodynamical pressure (5.17) resp. (5.17)' with $\mu_{110}(\xi,\zeta;\Omega) \equiv 0$ resp. $\mu_{110}(\xi,\zeta) \equiv 0$. On the other side, the distribution function $\sigma_{110}^{(0)}(\xi,\eta;\Omega)$ resp. $\sigma_{110}^{(0)}(\xi,\eta)$ is determined from (6.3) resp. (6.5) and consequently in this interesting special case we have the following explicit forms for the hydrodynamical pressure:

$$p_{110}(x,y,z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp[-i\Omega t]\Bigg[\int_{-\infty}^{\infty}\int_{-B}^{B}\Big[\int_{-\infty}^{\infty}\exp[i\Omega t]\{p_{100x}(\xi,\eta,-h,t)g_x(\xi,\eta)$$

$$+ p_{100y}(\xi,\eta,-h,t)g_y(\xi,\eta) + p_{100zz}(\xi,\eta,-h,t)g(\xi,\eta)\}dt\Big]$$

$$\cdot \bar{G}^{*(0)}(x,y,z;\xi,\eta;\Omega)\,d\xi\,d\eta\Bigg]d\Omega \qquad (6.07)$$

respectively

$$\bar{p}_{110}(\bar{x},\bar{y},\bar{z}) = \int_{-\infty}^{\infty}\int_{-B}^{B}\{\bar{p}_{100\bar{y}}(\xi,\eta,-h)g_{\bar{y}}(\eta) - \bar{p}_{100\bar{z}\bar{z}}(\xi,\eta,-h)g(\eta)\} \cdot \bar{G}^{(o)}(\bar{x},\bar{y},\bar{z};\xi,\eta;0)\,d\xi\,d\eta \tag{6.08}$$

For B = 0 (or in the unsymmetrical case $B_1 = \infty$ or $B_2 = \infty$ or both) corresponding formulas can be obtained; only we have to take the corresponding Green's functions and the corresponding pressure P_{100}.

6.2 The Distribution Functions of P_{101}

In the non-steady state case the boundary conditions (4.13) give, with $P_{101}(x,y,z,t)$ from (5.18)

$$\begin{aligned}\int_{-\infty}^{\infty} \exp[-i\Omega t]\,\sigma_{101}^{(i)}(x,z;\Omega)\,d\Omega = &\mp p_{100x}(x,\pm B_i,z,t)\,b_{ix}(x,z)\\ &\mp p_{100z}(x,\pm B_i,z,t)\,b_{iz}(x,z)\\ &\pm p_{100yy}(x,\pm B_i,z,t)\,b_i(x,z)\end{aligned}$$

$$-\infty < x < \infty \qquad -h < z < 0 \qquad i = 1,2$$

and the inversion formula of the Fourier transform yields

$$\begin{aligned}\sigma_{101}^{(i)}(x,z;\Omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp[i\Omega t]\{&\mp p_{100x}(x,\pm B_i,z,t)\,b_{ix}(x,z)\\ &\mp p_{100z}(x,\pm B_i,z,t)\,b_{iz}(x,z)\\ &\pm p_{100yy}(x,\pm B_i,z,t)\,b_i(x,z)\}\,dt\end{aligned}$$

$$-\infty < x < \infty \qquad -h < z < 0 \qquad i = 1,2. \tag{6.09}$$

Now the boundary condition (4.15) results in the following integral equation for $\mu_{101}(\xi,\zeta;\Omega)$:

$$\int_{-\infty}^{\infty} \exp[-i\Omega t]\Big[-\iint_{\bar{A}'}\{\mu_{101\xi}(\xi,\zeta;\Omega)\bar{G}_{\xi}(\bar{x}',0,\bar{z}';\xi,0,\zeta;\Omega)$$

$$+\mu_{101\zeta}(\xi,\zeta;\Omega)\bar{G}_{\zeta}(\bar{x}',0,\bar{z}';\xi,0,\zeta;\Omega)\}d\xi d\zeta$$

$$+\int_{\bar{C}'}\mu_{101}(\xi,\zeta;\Omega)\frac{\partial}{\partial n}\bar{G}(\bar{x}',0,\bar{z}';\xi,0,\zeta;\Omega)ds$$

$$+\int_{-\infty}^{\infty}\int_{-h}^{0}\{\sigma_{101}^{(1)}\bar{G}^{*(1)}(\bar{x}'+s_{000}t,0,\bar{z}';\xi,\zeta;\Omega)$$

$$+\sigma_{101}^{(2)}\bar{G}^{*(2)}(\bar{x}'+s_{000}t,0,\bar{z}';\xi,\zeta;\Omega)\}d\xi d\zeta\Big]d\Omega$$

$$= s_{000}\,\omega_{101}(t) - 2s_{000}\dot{\Theta}_{3101}(t) + \bar{x}'[\ddot{\Theta}_{3101}(t)+\dot{\omega}_{101}(t)] + (\bar{z}'-\bar{z}_c')\ddot{\Theta}_{1101}(t)$$

$$\bar{x}',\bar{z}' \text{ on } \bar{A}' \qquad (6.10)$$

In the steady-state case the boundary condition (4.15) gives with the hydrodynamical pressure $\bar{P}_{101}(\bar{x},\bar{y},\bar{z})$ from (5.18)' with respect to the moving coordinate system

$$\sigma_{101}^{(i)}(\bar{x},\bar{z}) = \mp\bar{\bar{P}}_{100\bar{z}}(\bar{x},\pm B_i,\bar{z})b_{i\bar{z}}(\bar{z}) \pm \bar{\bar{P}}_{100\bar{y}\bar{y}}(\bar{x},\pm B_i,\bar{z})\,b_i(\bar{z})$$

$$-\infty < \bar{x} < \infty$$
$$-h < \bar{z} < 0 \qquad i = 1,2 \qquad (6.11)$$

The distribution functions $\sigma_{101}^{(i)}(\bar{x},\bar{z})$ $i = 1, 2$ are therefore determined. Now the boundary condition (4.15) results in the following integral equation for $\mu_{101}(\xi,\zeta)$:

$$-\iint_{\bar{A}'}\{\mu_{101\xi}(\xi,\zeta)\,\bar{G}_\xi(\bar{x}',0,\bar{z}';\xi,0,\zeta;0)+\mu_{101\zeta}(\xi,\zeta)\,\bar{G}_\zeta(\bar{x}',0,\bar{z}';\xi,0,\zeta;0)\}\,d\xi\,d\zeta$$

$$+\int_{\bar{C}'}\mu_{101}(\xi,\zeta)\frac{\partial}{\partial n}\bar{G}(\bar{x}',0,\bar{z}';\xi,0,\zeta;0)\,ds$$

$$+\int_{-\infty}^{\infty}\int_{-h}^{0}\{\sigma_{101}^{(1)}(\xi,\zeta)\,\bar{G}^{(1)}(\bar{x}',0,\bar{z}';\xi,\zeta;0)+\sigma_{101}^{(2)}(\xi,\zeta)\,\bar{G}^{(2)}(\bar{x}',0,\bar{z}';\xi,\zeta;0)\}\,d\xi\,d\zeta=0$$

$$\bar{x}',\bar{z}' \quad \text{on } \bar{A}' \qquad (6.12)$$

Here also we used the result of section 7 that under the present conditions

$$\omega_{101}(t) = 0, \quad \Theta_{3101}(t) = \text{const.}, \quad \Theta_{1101}(t) = \text{const.}$$

When again $y = 0$ is a plane of symmetry of our canal the formulas reduce considerably. If

$$B_1 = B_2 = b \qquad \text{and} \qquad b_1(x,z) \equiv b_2(x,z) = b(x,z)$$

we have generally

$$\omega_{101}(t) = \Theta_{3101}(t) = \Theta_{1101}(t) = 0 ,$$

therefore the right-hand side of the boundary condition is zero. Because the function $P_{101}(x,y,zt)$ from (5.18) resp. $\bar{P}_{101}(\bar{x},\bar{y},\bar{z})$ from (5.18)' with $\mu_{101}(\xi,\zeta;\Omega) = 0$ resp. $\mu_{101}(\xi,\zeta) = 0$ become even functions of y resp. $\bar{y}$ the boundary condition (4.15) is evidently satisfied by the hydrodynamical pressure (5.18) resp. (5.18)' with $\mu_{101}(\xi,\zeta;\Omega) \equiv 0$ resp. $\mu_{101}(\xi,\zeta) \equiv 0$. Since on the other hand the distribution functions $\sigma_{101}^{(i)}(\xi,\zeta;\Omega)$ resp. $\sigma_{101}^{(i)}(\xi,\zeta)$ $i = 1,2$ are determined from (6.9) resp. (6.11) we have the following explicit forms for the hydrodynamical pressure:

$$p_{101}(x,y,z,t) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp[-i\Omega t]\cdot$$

$$\cdot\Big[\int_{-\infty}^{\infty}\int_{-h}^{0}\Big\{\int_{-\infty}^{\infty}\exp[i\Omega t]\,[p_{100x}(\xi,\mathfrak{z},\zeta,t)b_x(\xi,\zeta)+p_{100z}(\xi,\mathfrak{z},\zeta,t)b_z(\xi,\zeta)$$

$$-p_{100yy}(\xi,\mathfrak{z},\zeta,t)b(\xi,\zeta)]\,dt\Big\}\cdot$$

$$\cdot\{\bar{G}^{*(1)}(x,y,z;\xi,\zeta;\Omega)-\bar{G}^{(2)}(x,y,z;\xi,\zeta;\Omega)\}\,d\xi\,d\zeta\Big]\,d\Omega \qquad (6.13)$$

and

$$\bar{p}_{101}(\bar{x},\bar{y},\bar{z}) = -\int_{-\infty}^{\infty}\int_{-h}^{0}[\bar{p}_{100\bar{z}}(\xi,B,\zeta)b_{\bar{z}}(\zeta) - \bar{p}_{100\bar{y}\bar{y}}(\xi,B,\zeta)b(\zeta)]\cdot$$

$$\cdot\{\bar{G}^{(1)}(\bar{x},\bar{y},\bar{z};\xi,\zeta;0) - \bar{G}^{(1)}(\bar{x},\bar{y},\bar{z};\xi,\zeta;0)\}\,d\xi\,d\zeta \quad (6.14)$$

The special case of motion of the ship in a canal of infinite depth or along a wall can be handled in a similar way.

6.3 The Distribution Function for P_{111}

In the non-steady state case the boundary condition (4.17) gives, with $P_{111}(x,y,z,t)$ from (5.19)

$$\int_{-\infty}^{\infty}\exp[-i\Omega t]\,\sigma_{111}^{(0)}(x,y;\Omega)\,d\Omega = p_{101x}(x,y,-h,t)g_x(x,y) + p_{101y}(x,y,-h,t)g_y(x,y) - p_{101zz}(x,y,-h,t)g(x,y) \qquad \begin{matrix}-\infty<x<\infty;\\ -B_2<y<B_1\end{matrix} \quad (6.15)$$

and by the inversion formula of the Fourier transform:

$$\sigma_{111}^{(0)}(x,y;\Omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp[i\Omega t]\{p_{101x}(x,y,-h,t)\,g_x(x,y) + p_{101y}(x,y,-h,t)\,g_y(x,y) - p_{101zz}(x,y,-h,t)\,g(x,y)\}\,dt$$

$$-\infty < x < \infty$$

$$-B_2 < y < B_1 \quad (6.16)$$

Analogously the boundary conditions (4.18) give with the same $P_{111}(x,y,z,t)$ the following equations for the distribution functions $\sigma_{111}^{(i)}(x,z;\Omega)$ $i = 1,2$:

$$\int_{-\infty}^{\infty} \exp[-i\Omega t]\,\sigma_{111}^{(i)}(x,z;\Omega)\,d\Omega = \mp P_{110x}(x,\pm B_i,z,t)\,b_{ix}(x,z)$$
$$\mp P_{110z}(x,\pm B_i,z,t)\,b_{iz}(x,z)$$
$$\pm P_{110yy}(x,\pm B_i,z,t)\,b_i(x,z)$$
$$-\infty < x < \infty \qquad i = 1,2 \qquad (6.17)$$
$$-h < z < 0$$

and by the inversion formula of the Fourier transform

$$\sigma_{111}^{(i)}(x,z;\Omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp[i\Omega t]\{\mp P_{110x}(x,\pm B_i,z,t)\,b_{ix}(x,z)$$
$$\mp P_{110z}(x,\pm B_i,z,t)\,b_{iz}(x,z)$$
$$\pm P_{100yy}(x,\pm B_i,z,t)\,b_i(x,z)\}\,dt$$
$$-\infty < x < \infty \qquad i = 1,2\,. \qquad (6.18)$$
$$-h < z < 0$$

By (6.16) and (6.18) the distribution functions

$$\sigma_{111}^{(0)}(x,y;\Omega)\,,\quad \sigma_{111}^{(1)}(x,z;\Omega)\,,\quad \sigma_{111}^{(2)}(x,z;\Omega)$$

are explicitely known provided the functions $P_{110}(x,y,z,t)$ and $P_{101}(x,y,z,t)$ are determined in the preceding steps. Now the boundary condition (4.20) results in the following integral equation for $\mu_{111}(\xi,\zeta;\Omega)$:

$$\int_{-\infty}^{\infty} \exp[-i\Omega t]\Big[-\iint_{\bar A'}\{\mu_{111\xi}(\xi,\zeta;\Omega)\,\bar G_{\xi}(\bar x',0,\bar z';\xi,0,\zeta;\Omega)$$

$$+\mu_{111\zeta}(\xi,\zeta;\Omega)\,\bar G_{\zeta}(\bar x',0,\bar z';\xi,0,\zeta;\Omega)\}d\xi d\zeta$$

$$+\int_{\bar C'}\mu_{111}(\xi,\zeta;\Omega)\frac{\partial}{\partial n}\bar G(\bar x',0,\bar z';\xi,0,\zeta;\Omega)\,ds$$

$$+\int_{-\infty}^{\infty}\int_{-B_2}^{B_1}\sigma^{(0)}_{111}(\xi,\eta;\Omega)\,\bar G^{*(0)}(\bar x'+s_{000}t,0,\bar z';\xi,\eta;\Omega)\,d\xi d\eta$$

$$+\int_{-\infty}^{\infty}\int_{-h}^{0}\{\sigma^{(1)}_{111}(\xi,\zeta;\Omega)\bar G^{*(1)}(\bar x'+s_{000}t,0,\bar z';\xi,\zeta;\Omega)$$

$$+\sigma^{(2)}_{111}(\xi,\zeta;\Omega)\bar G^{*(2)}(\bar x'+s_{000}t,0,\bar z';\xi,\zeta;\Omega)\}d\xi d\zeta\Big]d\Omega$$

$$= s_{000}\,\omega_{111}(t) - 2s_{000}\,\dot\Theta_{3111}(t) + \bar x'[\ddot\Theta_{3111}(t)+\dot\omega_{111}(t)] + (\bar z'-\bar z_c')\,\ddot\Theta_{1111}(t)$$

$$\bar x',\bar z' \text{ on } \bar A' \quad . \qquad (6.19)$$

In the steady-state case the boundary condition (4.19) gives, with the hydrodynamical pressure $\bar P_{111}(\bar x,\bar y,\bar z)$ from (5.19)' with respect to the moving coordinate system

$$\sigma^{(0)}_{111}(\bar x,\bar y) = \bar p_{101\bar y}(\bar x,\bar y,-h)\,g_{\bar y}(\bar y) - \bar p_{101}(\bar x,\bar y,-h)\,g(\bar y)$$

$$-\infty < \bar x < \infty$$

$$-B_2 < \bar y < B_1 \qquad (6.20)$$

and the boundary conditions (4.18) yield

$$\sigma^{(i)}_{111}(\bar{x},\bar{z}) = \mp \bar{p}_{110\bar{z}}(\bar{x},\pm\mathfrak{B}_i,\bar{z})\, b_{i\bar{z}}(\bar{z}) \pm \bar{p}_{110\bar{y}\bar{y}}(\bar{x},\pm\mathfrak{B}_i,\bar{z})\, b_i(\bar{z})$$

$$-\infty < \bar{x} < \infty \qquad i = 1,2 \quad . \qquad (6.21)$$
$$-h < \bar{z} < 0$$

By (2.20) and (2.21) the distribution functions $\sigma^{(0)}_{111}(\bar{x},\bar{y})$ and $\sigma^{(i)}_{111}(\bar{x},\bar{z})$ $i = 1,2$ are determined, provided we know the pressures $P_{110}(\bar{x},\bar{y},\bar{z})$ and $\bar{P}_{101}(\bar{x},\bar{y},\bar{z})$ from the preceding steps. The boundary condition (4.20) on the ship's hull now yields the following integral equation for $\mu_{111}(\xi,\zeta)$:

$$-\iint_{\bar{A}'}\left\{\mu_{111\xi}(\xi,\zeta)\bar{G}_{\xi}(\bar{x}',0,\bar{z}';\xi,0,\zeta;0) + \mu_{111\zeta}(\xi,\zeta)\bar{G}_{\zeta}(\bar{x}',0,\bar{z}';\xi,0,\zeta;0)\right\}d\xi\, d\zeta$$

$$+\int_{\bar{C}'}\mu_{111}(\xi,\zeta)\frac{\partial}{\partial n}\bar{G}(\bar{x}',0,\bar{z}';\xi,0,\zeta;0)\,ds$$

$$+\int_{-\infty}^{\infty}\int_{-\mathfrak{B}_2}^{\mathfrak{B}_1}\sigma^{(0)}_{111}(\xi,\eta)\,\bar{G}^{(0)}(\bar{x}',0,\bar{z}';\xi,\eta;0)\,d\xi\, d\eta$$

$$+\int_{-\infty}^{\infty}\int_{-h}^{0}\left\{\sigma^{(1)}_{111}(\xi,\zeta)\,\bar{G}^{(1)}(\bar{x}',0,\bar{z}';\xi,\zeta;0) + \sigma^{(2)}_{111}(\xi,\zeta)\bar{G}^{(2)}(\bar{x}',0,\bar{z}';\xi,\zeta;0)\right\}d\xi\, d\zeta$$

$$= 0 \qquad\qquad \bar{x}',\ \bar{z}' \text{ on } \bar{A}' \quad . \qquad (6.22)$$

Here we use the result of section 7 that under the present conditions

$$\omega_{111}(t) = 0\ , \quad \theta_{3111}(t) = \text{const.}\ , \quad \theta_{1111}(t) = \text{const.}$$

The formulas become extremely simple, if y = 0 is a plane of symmetry of our canal. Then

$$\mathfrak{B}_1 = \mathfrak{B}_2 = \mathfrak{B}\ , \qquad b_1(x,z) \equiv b_2(x,z) = b(x,z)\ ,$$
$$g(x,-y) = g(x,y) \qquad (6.23)$$

and we have generally (also in the non-steady state case)

$$\omega_{111}(t) = \theta_{3111}(t) \;=\; \theta_{1111}(t) = 0 \;;$$

therefore the right hand side of the boundary condition (4.20) is zero. Because under conditions (2.23) the function $P_{111}(x,y,z,t)$ from (5.19) resp. $\overline{P}_{111}(\overline{x},\overline{y},\overline{z})$ from (5.19)' with $\mu_{111}(\xi,\zeta;\Omega) \equiv 0$ resp. $\mu_{111}(\xi,\zeta) \equiv 0$ become even functions of y resp. $\overline{y}$ the boundary condition (4.20) is evidently satisfied by the hydrodynamical pressure (5.18) resp. (5.18)' with $\mu_{111}(\xi,\zeta;\Omega) \equiv 0$ resp. $\mu_{111}(\xi,\zeta,) = 0$. On the other hand the distribution functions

$$\sigma_{111}^{(0)}(\xi,\eta;\Omega),\quad \sigma_{111}^{(i)}(\xi,\zeta;\Omega) \quad \text{resp.} \quad \sigma_{111}^{(0)}(\xi,\eta),\quad \sigma_{111}^{(i)}(\xi,\zeta)$$

$$i = 1,2$$

are determined from (6.16), (6.18) resp. (6.20), (6.21). Therefore we have the following explicit forms for the hydrodynamical pressure:

$$\begin{aligned} p_{111}(x,y,z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp[-i\Omega t]\Bigg[\int_{-\infty}^{\infty}\int_{-B}^{B} \Big\{ \int_{-\infty}^{\infty} \exp[i\Omega t]\big[& p_{101x}(\xi,\eta,-h,t)\, g_x(\xi,\eta) \\ & + p_{101y}(\xi,\eta,-h,t)\, g_y(\xi,\eta) \\ & - p_{101zz}(\xi,\eta,-h,t)\, g(\xi,\eta)\big] dt \Big\} \\ & \cdot \bar{G}^{*(0)}(x,y,z;\xi,\eta;\Omega)\, d\xi\, d\eta \\ - \int_{-\infty}^{\infty}\int_{-h}^{0} \Big\{ \int_{-\infty}^{\infty} \exp[i\Omega t]\big[& p_{110x}(\xi,B,\zeta,t)\, b_x(\xi,\zeta) \\ & + p_{110z}(\xi,B,\zeta,t)\, b_z(\xi,\zeta) \\ & - p_{110yy}(\xi,B,\zeta,t)\, b(\xi,\zeta)\big] dt \Big\} \\ & \cdot \Big\{ \bar{G}^{*(1)}(x,y,z;\xi,\zeta;\Omega) - \bar{G}^{*(2)}(x,y,z;\xi,\zeta;\Omega) \Big\} d\xi\, d\zeta \Bigg] d\Omega \end{aligned} \qquad (6.24)$$

respectively

$$\bar{\varphi}_{111}(\bar{x},\bar{y},\bar{z}) = \int_{-\infty}^{\infty}\int_{-B}^{B}\{\bar{\varphi}_{101\bar{y}}(\xi,\eta,-h)\,q_{\bar{y}}(\eta) - \bar{\varphi}_{101\bar{z}\bar{z}}(\xi,\eta,-h)\,q(\eta)\} \cdot \bar{G}^{(0)}(\bar{x},\bar{y},\bar{z};\xi,\eta;0)\,d\xi\,d\eta$$

$$-\int_{-\infty}^{\infty}\int_{-h}^{0}\{\bar{\varphi}_{110\bar{z}}(\xi,B,\zeta)\,b_{\bar{z}}(\zeta) - \bar{\varphi}_{110\bar{y}\bar{y}}(\xi,B,\zeta)\,b(\zeta)\} \cdot \{\bar{G}^{(1)}(\bar{x},\bar{y},\bar{z};\xi,\zeta;0) - \bar{G}^{(2)}(\bar{x},\bar{y},\bar{z};\xi,\zeta;0)\}\,d\xi\,d\zeta \quad (6.25)$$

These are the formulas which we intended to give here. A numerical calculation especially for the steady-state cases will be treated in a future publication. There also the wave resistance in canals of various cross sections will be examined.

7. Equations of Motion of the Ship

On the right hand side of the boundary conditions (3.19) on the ship's hull there appear the physical variables $\omega_{1jk}(t)$, $\Theta_{31jk}(t)$ and $\Theta_{11jk}(t)$ $j,k = 0,1$ which at first we suppose to be given. But really these terms just as the other terms $S_{1jk}(t)$, $z_{1jk}(t)$ and $\Theta_{21jk}(t)$ which fix the motion of the ship as a floating rigid body under the action of its weight $M \cdot g$, the propeller thrust $\mathfrak{T}$, rudder force R and the pressure p of the water on its hull must satisfy some differential equations, the differential equations of the motion of the ship, which will be derived in this section. First, the principle of the motion of the center of gravity yields the condition

$$M\frac{d}{dt}(s\,n_{\bar{x}} + \dot{\bar{z}}_c n_{\bar{z}}) = \int_S p\,n\,dS + \mathfrak{T} + R - Mg\,n_{\bar{z}}. \quad (7.01)$$

By n we mean the inward normal on the hull. Now by (2.6) we have

$$\frac{d}{dt}n_{\bar{x}} = -\dot{\alpha}\{n_x \sin\alpha + n_y \cos\alpha\} = -\dot{\alpha}\,n_{\bar{y}} = -\omega\cdot n_{\bar{y}}, \quad (7.02)$$

$$\frac{d}{dt}\mathfrak{r}_{\bar{y}} = \dot{\alpha}\{\mathfrak{r}_x\cos\alpha - \mathfrak{r}_y\sin\alpha\} = \dot{\alpha}\,\mathfrak{r}_{\bar{x}} = \omega\cdot\mathfrak{r}_{\bar{x}}\,, \tag{7.03}$$

$$\frac{d}{dt}\mathfrak{r}_{\bar{z}} = \frac{d}{dt}\mathfrak{r}_z = 0\,, \tag{7.04}$$

because the unit vectors $\mathfrak{r}_x, \mathfrak{r}_y, \mathfrak{r}_z$ are independent of time t. Therefore (7.1) can be written in the form

$$M\dot{s}\,\mathfrak{r}_{\bar{x}} - Ms\omega\,\mathfrak{r}_{\bar{y}} + M\ddot{z}_c\,\mathfrak{r}_z = \int_S p\,\mathfrak{n}\,dS + \mathfrak{T} + \mathfrak{R} - Mg\,\mathfrak{r}_z \tag{7.05}$$

with the pressure p defined by Bernoulli's law (1.5). The law of conservation of angular momentum is written with respect to the center of gravity and takes the form

$$\frac{d}{dt}\int_M(\mathfrak{r} - \mathfrak{r}_c)\times(\dot{\mathfrak{r}} - \dot{\mathfrak{r}}_c)\,dm = \int_S p(\mathfrak{r} - \mathfrak{r}_c)\times\mathfrak{n}\,dS + (\mathfrak{r}_T - \mathfrak{r}_c)\times\mathfrak{T} + (\mathfrak{r}_R - \mathfrak{r}_c)\times\mathfrak{R}\,. \tag{7.06}$$

By $\mathfrak{r} = \{x, y, z\} = x\,\mathfrak{r}_x + y\,\mathfrak{r}_y + z\,\mathfrak{r}_z$ is meant the position vector of the element of mass dm relative to the fixed coordinate system; $\mathfrak{r}_c$ fixes the position of the center of gravity, $\mathfrak{r}_T$, $\mathfrak{r}_R$ locate the points of application of the propeller thrust $\mathfrak{T}$ and rudder force R respectively. The crosses indicate vector products both here and later on. If we introduce the position vector

$$\bar{\mathfrak{r}}' = \{\bar{x}', \bar{y}', \bar{z}'\} = \bar{x}'\mathfrak{r}_{\bar{x}'} + \bar{y}'\mathfrak{r}_{\bar{y}'} + \bar{z}'\mathfrak{r}_{\bar{z}'} \tag{7.07}$$

with respect to the $\bar{x}'$, $\bar{y}'$, $\bar{z}'$-coordinate system fixed in the ship, we obtain

$$\bar{\mathfrak{q}} = \bar{\mathfrak{r}}' - \bar{z}_c'\cdot\mathfrak{r}_{\bar{z}'} \tag{7.08}$$

for the vector $\bar{\mathfrak{q}}$ from the center of gravity to any other point in the ship.

Because the motion of a rigid body can be decomposed in the translation of point fixed in the body, for instance the center of gravity, and a rotation of the body with respect to an axes through this point we have the relation

$$\dot{\mathbf{r}} = \dot{\mathbf{r}}_c - (\vec{\omega} + \dot{\vartheta}) \times \bar{\mathbf{q}} \qquad \vec{\omega} = \omega \mathbf{n}_z \tag{7.09}$$

since $\vec{\omega} + \dot{\vartheta}$ is the angular velocity of the ship. (cf. (2.3) and (2.19)). Therefore with respect to the moving $\bar{x}'$, $\bar{y}'$, $\bar{z}'$-coordinate system the dynamical condition (7.6) yields

$$-\frac{d}{dt}\int_M (\bar{\mathbf{r}}' - \bar{z}'_c \mathbf{n}_{\bar{z}'}) \times [(\vec{\omega} + \dot{\vartheta}) \times (\bar{\mathbf{r}}' - \bar{z}'_c \mathbf{n}_{\bar{z}'})]\, dm$$
$$= \int_S p(\bar{\mathbf{r}}' - \bar{z}'_c \mathbf{n}_{\bar{z}'}) \times \mathbf{n}\, dS + (\mathbf{r}_q - \mathbf{r}_c) \times \mathbf{q} + (\mathbf{r}_R - \mathbf{r}_c) \times \mathbf{R}\,. \tag{7.10}$$

Equations (7.5) and (7.10) must now be developed with respect to the parameters ϵ_1, ϵ_2, ϵ_3.

We begin with the non-integrated terms in Equations (7.5) and (7.10), which can be obtained easily once the developments for the mass M of the ship, the propeller thrust $\underline{\mathbf{q}}$ and the rudder force R are given.

Since the volume of our ship is of order ϵ_1, the mass should also be assumed to be of order ϵ_1:

$$M = \epsilon_1 M_{100} \tag{7.11}$$

The propeller thrust $\mathbf{q}$ is in the $\bar{x}'$-direction. Since we assumed that the displacements are small relative to a motion with constant velocity, the acceleration of the ship in the $\bar{x}'$-direction must be small and it is clear that following expansion for $\mathbf{q}$ should be assumed

$$\mathbf{q} = [\epsilon_1^2 T_{200} + \epsilon_1^2 \epsilon_2 T_{210} + \epsilon_1^2 \epsilon_3 T_{201} + \epsilon_1^2 \epsilon_2 \epsilon_3 T_{211} + O(\epsilon_1^2)]\, \mathbf{n}_{\bar{x}'}$$

From (2.26) we then obtain with respect to the $\bar{x}$, $\bar{y}$, $\bar{z}$-coordinate system the expansion:

$$\mathbf{q} = [\epsilon_1^2 T_{200} + \epsilon_1^2 \epsilon_2 T_{210} + \epsilon_1^2 \epsilon_3 T_{201} + \epsilon_1^2 \epsilon_2 \epsilon_3 T_{211} + O(\epsilon_1^2)]\, \mathbf{n}_{\bar{x}} + \mathbf{n}_{\bar{y}} O(\epsilon_1^3) + \mathbf{n}_{\bar{z}} O(\epsilon_1^3) \tag{7.12}$$

For the rudder force R we assume with respect to the $\bar{x}'$, $\bar{y}'$, $\bar{z}'$-coordinate system the expansion:

$$
\begin{aligned}
R = n_{\bar{x}'}&\{\varepsilon_1^2 R_{1200} + \varepsilon_1^2\varepsilon_2 R_{1210} + \varepsilon_1^2\varepsilon_3 R_{1201} + \varepsilon_1^2\varepsilon_2\varepsilon_3 R_{1211} + \cdots\} \\
+ n_{\bar{y}'}&\{\varepsilon_1 R_{2100} + \varepsilon_1\varepsilon_2 R_{2110} + \varepsilon_1\varepsilon_3 R_{2101} + \varepsilon_1\varepsilon_2\varepsilon_3 R_{2111} + \cdots\} \\
+ n_{\bar{z}'}&\{\varepsilon_1^2 R_{3200} + \varepsilon_1^2\varepsilon_2 R_{3210} + \varepsilon_1^2\varepsilon_3 R_{3201} + \varepsilon_1^2\varepsilon_2\varepsilon_3 R_{3211} + \cdots\} \qquad (7.13)
\end{aligned}
$$

With respect to the $\bar{x}$, $\bar{y}$, $\bar{z}$-coordinate system we obtain from (2.26) the expansion:

$$
\begin{aligned}
R = n_{\bar{x}}\{&\varepsilon_1^2(R_{1200} + R_{2100}\Theta_{3100}) + \varepsilon_1^2\varepsilon_2(R_{1210} + R_{2100}\Theta_{3110} + R_{2110}\Theta_{3100}) \\
&+ \varepsilon_1^2\varepsilon_3(R_{1201} + R_{2100}\Theta_{3101} + R_{2101}\Theta_{3100}) \\
&+ \varepsilon_1^2\varepsilon_2\varepsilon_3(R_{1211} + R_{2100}\Theta_{3111} + R_{2111}\Theta_{3100} + R_{2110}\Theta_{3101} + R_{2101}\Theta_{3110}) + \cdots\} \\
+ n_{\bar{y}}\{&\varepsilon_1 R_{2100} + \varepsilon_1\varepsilon_2 R_{2110} + \varepsilon_1\varepsilon_3 R_{2101} + \varepsilon_1\varepsilon_2\varepsilon_3 R_{2111} + \cdots\} \\
+ n_{\bar{z}}\{&\varepsilon_1^2(R_{3200} - R_{2100}\Theta_{1100}) + \varepsilon_1^2\varepsilon_2(R_{3210} - R_{2100}\Theta_{1110} + R_{2110}\Theta_{1100}) \\
&+ \varepsilon_1^2\varepsilon_3(R_{3201} - R_{2100}\Theta_{1101} - R_{2101}\Theta_{1100}) \\
&+ \varepsilon_1^2\varepsilon_2\varepsilon_3(R_{3211} - R_{2100}\Theta_{1111} - R_{2111}\Theta_{1100} - R_{2110}\Theta_{1101} - R_{2101}\Theta_{1110}) + \cdots\} \qquad (7.14)
\end{aligned}
$$

Substituting (7.11), (7.12) and (7.14) together with the expansions (2.11), (2.12), (2.23) and noting that $\omega_{000} = 0$, (7.5) takes the form:

$$
\begin{aligned}
&n_{\bar{x}}\{\varepsilon_1 M_{100}\dot{s}_{000} + \varepsilon_1^2 M_{100}\dot{s}_{100} + \varepsilon_1^2\varepsilon_2 M_{100}\dot{s}_{110} + \varepsilon_1^2\varepsilon_3 M_{100}\dot{s}_{101} + \varepsilon_1^2\varepsilon_2\varepsilon_3 M_{100}\dot{s}_{111} + \cdots\} \\
&- n_{\bar{y}}\{\varepsilon_1^2 M_{100}s_{000}\omega_{100} + \varepsilon_1^2\varepsilon_2 M_{100}s_{000}\omega_{110} + \varepsilon_1^2\varepsilon_3 M_{100}s_{000}\omega_{101} + \varepsilon_1^2\varepsilon_2\varepsilon_3 M_{100}s_{000}\omega_{111} + \cdots\} \\
&+ n_{\bar{z}}\{\varepsilon_1^2 M_{100}\ddot{z}_{100} + \varepsilon_1^2\varepsilon_2 M_{100}\ddot{z}_{110} + \varepsilon_1^2\varepsilon_3 M_{100}\ddot{z}_{101} + \varepsilon_1^2\varepsilon_2\varepsilon_3 M_{100}\ddot{z}_{111} + \cdots\} \\
&= \int_S p\, n\, dS + n_{\bar{x}}\{\varepsilon_1^2(T_{200} + R_{1200} + R_{2100}\Theta_{3100}) + \varepsilon_1^2\varepsilon_2(T_{210} + R_{1210} + R_{2100}\Theta_{3110} + R_{2110}\Theta_{3100}) \\
&\qquad + \varepsilon_1^2\varepsilon_3(T_{201} + R_{1201} + R_{2100}\Theta_{3101} + R_{2101}\Theta_{3100}) \\
&\qquad + \varepsilon_1^2\varepsilon_2\varepsilon_3(T_{211} + R_{1211} + R_{2100}\Theta_{3111} + R_{2111}\Theta_{3100} + R_{2110}\Theta_{3101} + R_{2101}\Theta_{3110}) + \cdots\}
\end{aligned}
$$

$$\mathbf{i}_{\bar{y}}\left\{\epsilon_1 R_{2100}+\epsilon_1\epsilon_2 R_{2110}+\epsilon_1\epsilon_3 R_{2101}+\epsilon_1\epsilon_2\epsilon_3 R_{2111}+\cdots\right\}$$

$$+\mathbf{i}_{\bar{z}}\Big\{-\epsilon_1 M_{100}\,g+\epsilon_1^2(R_{3200}-R_{2100}\Theta_{1100})+\epsilon_1^2\epsilon_2(R_{3210}-R_{2100}\Theta_{1110}-R_{2110}\Theta_{1100})$$
$$+\epsilon_1^2\epsilon_3(R_{3201}-R_{2100}\Theta_{1101}-R_{2101}\Theta_{1100})$$
$$+\epsilon_1^2\epsilon_2\epsilon_3(R_{3211}-R_{2100}\Theta_{1111}-R_{2111}\Theta_{1100}-R_{2110}\Theta_{1101}-R_{2101}\Theta_{1110})+\cdots\Big\} \quad (7.15)$$

The equation (7.10) for the conservation of angular momentum can be expanded analogously in a power series of ϵ_1, ϵ_2, ϵ_3. We assume that the propeller thrust $\mathbf{T}$ is applied at a point in the vertical plane of symmetry of the ship with coordinates $(-\bar{\ell}'_t, 0, -\bar{t}'_t - \bar{z}'_c)$ with respect to the $\bar{x}'$, $\bar{y}'$, $\bar{z}'$-coordinate system fixed in the ship. It follows that

$$(\mathbf{r}_T-\mathbf{r}_c)\times\mathbf{T} = (-\bar{\ell}'_t\,\mathbf{i}_{\bar{x}'}-\bar{t}'_t\,\mathbf{i}_{\bar{z}'})\times\mathbf{T}$$
$$= -(\epsilon_1^2 T_{200}+\epsilon_1^2\epsilon_2 T_{210}+\epsilon_1^2\epsilon_3 T_{201}+\epsilon_1^2\epsilon_2\epsilon_3 T_{211}+\cdots)$$
$$\cdot(\bar{\ell}'_t\,\mathbf{i}_{\bar{x}'}+\bar{t}'_t\,\mathbf{i}_{\bar{z}'})\times\mathbf{i}_{\bar{x}'}$$
$$= -\bar{t}'_t(\epsilon_1^2 T_{200}+\epsilon_1^2\epsilon_2 T_{210}+\epsilon_1^2\epsilon_3 T_{201}+\epsilon_1^2\epsilon_2\epsilon_3 T_{211}+\cdots)\,\mathbf{i}_{\bar{y}'}$$
$$= -\bar{t}'_t(\epsilon_1^2 T_{200}+\epsilon_1^2\epsilon_2 T_{210}+\epsilon_1^2\epsilon_3 T_{201}+\epsilon_1^2\epsilon_2\epsilon_3 T_{211}+\cdots)\,\mathbf{i}_{\bar{y}}$$
$$+\,\mathbf{i}_{\bar{x}}\,O(\epsilon_1^3)+\mathbf{i}_{\bar{z}}\,O(\epsilon_1^3) \quad . \quad (7.18)$$

Now we must derive the corresponding development for the term $(\mathbf{r}_R-\mathbf{r}_c)\times\mathbf{R}$. In accordance with (7.13) the rudder force R is the resultant of rudder forces R_{ijk}:

$$\mathbf{R} = \sum_{i=1,j,k=0}^{\infty}\epsilon_1^i\epsilon_2^j\epsilon_3^k\,\mathbf{R}_{ijk}$$

with
$$\mathbf{R}_{ijk} = R_{1ijk}\,\mathbf{i}_{\bar{x}'}+R_{2ijk}\,\mathbf{i}_{\bar{y}'}+R_{3ijk}\,\mathbf{i}_{\bar{z}'}$$

and
$$R_{11jk} = R_{31jk} = 0 \qquad \text{for } j,k = 0,1 \;.$$

If now $\mathbf{r}_R$ is the position vector of R and $\mathbf{r}_{R_{ijk}}$ the position vectors of R_{ijk} the following equation holds by the definition of the resultant force of a system of forces:

$$\mathbf{r}_R \times \mathbf{R} = \sum_{i=1,\,j,k=0}^{\infty} \varepsilon_1^i \varepsilon_2^j \varepsilon_3^k \; \mathbf{r}_{R_{ijk}} \times \mathbf{R}_{ijk} \quad .$$

It follows that

$$(\mathbf{r}_R - \mathbf{r}_c) \times \mathbf{R} = \sum_{i=1,\,j,k=0}^{\infty} \varepsilon_1^i \varepsilon_2^j \varepsilon_3^k \,(\mathbf{r}_{R_{ijk}} - \mathbf{r}_c) \times \mathbf{R}_{ijk} \, .$$

We assume that the rudder force R_{ijk} is applied at a point in the vertical plane of symmetry of the ship with coordinates $(-\bar{l}_{rijk}, 0, \bar{t}'_{rijk}-\bar{z}'_c)$ with respect to the $\bar{x}'$, $\bar{y}'$, $\bar{z}'$-coordinate system. Then

$$\mathbf{r}_{R_{ijk}} - \mathbf{r}_c = - \bar{l}'_{rijk}\, \mathbf{n}_{\bar{x}'} - \bar{t}'_{rijk}\, \mathbf{n}_{\bar{z}'} \tag{7.19}$$

and we have

$$(\mathbf{r}_R - \mathbf{r}_c) \times \mathbf{R} = - \sum_{i=1,\,j,k=0}^{\infty} \varepsilon_1^i \varepsilon_2^j \varepsilon_3^k (\bar{l}'_{rijk}\mathbf{n}_{\bar{x}'} + \bar{t}'_{rijk}\mathbf{n}_{\bar{z}'}) \times (R_{1ijk}\mathbf{n}_{\bar{x}'} + R_{2ijk}\mathbf{n}_{\bar{y}'} + R_{3ijk}\mathbf{n}_{\bar{z}'})$$

$$= \sum_{i=1,\,j,k=0}^{\infty} \varepsilon_1^i \varepsilon_2^j \varepsilon_3^k \{ \mathbf{n}_{\bar{x}'} (\bar{t}'_{rijk} R_{2ijk})$$

$$- \mathbf{n}_{\bar{y}'} (\bar{t}'_{rijk} R_{1ijk} - \bar{l}'_{rijk} R_{3ijk}) - \mathbf{n}_{\bar{z}} (\bar{l}'_{rijk} R_{2ijk}) \}$$

$$= \mathbf{n}_{\bar{x}'} \Big\{ \sum_{i=1,\,j,k=0}^{\infty} \varepsilon_1^i \varepsilon_2^j \varepsilon_3^k \, \bar{t}'_{rijk} R_{2ijk} \Big\}$$

$$- \mathbf{n}_{\bar{y}'} \Big\{ \sum_{i=1,\,j,k=0}^{\infty} \varepsilon_1^i \varepsilon_2^j \varepsilon_3^k \, (\bar{t}'_{rijk} R_{1ijk} - \bar{l}'_{rijk} R_{3ijk}) \Big\}$$

$$- \mathbf{n}_{\bar{z}'} \Big\{ \sum_{i=1,\,j,k=0}^{\infty} \varepsilon_1^i \varepsilon_2^j \varepsilon_3^k \, \bar{l}'_{rijk} R_{2ijk} \Big\}$$

$$= \mathbf{n}_{\bar{x}'} \{ \varepsilon_1 \bar{t}'_{r100} R_{2100} + \varepsilon_1\varepsilon_2 \bar{t}'_{r110} R_{2110} + \varepsilon_1\varepsilon_3 \bar{t}'_{r101} R_{2101} + \bar{t}'_{r111} R_{2111} + \cdots \}$$

$$- \mathbf{n}_{\bar{y}'} \{ \varepsilon_1^2 (\bar{t}'_{r200} R_{1200} - \bar{l}'_{r200} R_{3200}) + \varepsilon_1^2\varepsilon_2 (\bar{t}'_{r210} R_{1210} - \bar{l}'_{r210} R_{3210})$$

$$+ \varepsilon_1^2\varepsilon_3 (\bar{t}'_{r201} R_{1201} - \bar{l}'_{r201} R_{3201}) + \varepsilon_1^2\varepsilon_2\varepsilon_3 (\bar{t}'_{r211} R_{1211} - \bar{l}'_{r211} R_{3211}) + \cdots \}$$

$$- \mathbf{n}_{\bar{z}'} \{ \varepsilon_1 \bar{l}'_{r100} R_{2100} + \varepsilon_1\varepsilon_2 \bar{l}'_{r110} R_{2110} + \varepsilon_1\varepsilon_3 \bar{l}'_{r101} R_{2101} + \varepsilon_1\varepsilon_2\varepsilon_3 \bar{l}'_{r111} R_{2111} + \cdots \}$$

$$= \mathbf{n}_{\bar{x}} \{ \varepsilon_1 \bar{t}'_{r100} R_{2100} + \varepsilon_1\varepsilon_2 \bar{t}'_{r110} R_{2110} + \varepsilon_1\varepsilon_3 \bar{t}'_{r101} R_{2101} + \varepsilon_1\varepsilon_2\varepsilon_3 \bar{t}'_{r111} R_{2111} + \cdots \}$$

$$- \mathbf{n}_{\bar{y}} \{ \varepsilon_1^2 [(\bar{t}'_{r200} R_{1200} + \bar{t}'_{r100} R_{2100} \Theta_{3100}) - (\bar{l}'_{r200} R_{3200} - \bar{l}'_{r100} R_{2100} \Theta_{1100})]$$

$$+\epsilon_1^2\epsilon_2\Big[\big(\bar{t}'_{\tau 210}R_{1210}+\bar{t}'_{\tau 110}R_{2110}\Theta_{3100}+\bar{t}'_{\tau 100}R_{2100}\Theta_{3110}\big)$$
$$-\big(\bar{\ell}'_{\tau 210}R_{3210}-\bar{\ell}'_{\tau 110}R_{2110}\Theta_{1100}-\bar{\ell}'_{\tau 100}R_{2100}\Theta_{1110}\big)\Big]$$
$$+\epsilon_1^2\epsilon_3\Big[\big(\bar{t}'_{\tau 201}R_{1201}+\bar{t}'_{\tau 101}R_{2101}\Theta_{3100}+\bar{t}'_{\tau 100}R_{2100}\Theta_{3101}\big)$$
$$-\big(\bar{\ell}'_{\tau 201}R_{3201}-\bar{\ell}'_{\tau 101}R_{2101}\Theta_{1100}-\bar{\ell}'_{\tau 100}R_{2100}\Theta_{1101}\big)\Big]$$
$$+\epsilon_1^2\epsilon_2\epsilon_3\Big[\big(\bar{t}'_{\tau 211}R_{1211}+\bar{t}'_{\tau 110}R_{2110}\Theta_{3101}+\bar{t}'_{\tau 101}R_{2101}\Theta_{3110}+\bar{t}'_{\tau 100}R_{2100}\Theta_{3111}+\bar{t}'_{\tau 111}R_{2111}\Theta_{3100}\big)$$
$$-\big(\bar{\ell}'_{\tau 211}R_{3211}-\bar{\ell}'_{\tau 110}R_{2110}\Theta_{1101}-\bar{\ell}'_{\tau 101}R_{2101}\Theta_{1110}-\bar{\ell}'_{\tau 100}R_{2100}\Theta_{1111}-\bar{\ell}'_{\tau 111}R_{2111}\Theta_{1100}\Big]$$
$$+\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\Big\}$$
$$-\varkappa_{\bar{z}}\Big\{\epsilon_1\bar{\ell}'_{\tau 100}R_{2100}+\epsilon_1\epsilon_2\bar{\ell}'_{\tau 110}R_{2110}+\epsilon_1\epsilon_3\bar{\ell}'_{\tau 101}R_{2101}+\epsilon_1\epsilon_2\epsilon_3\bar{\ell}'_{\tau 111}R_{2111}+\cdots\Big\} \quad (7.20)$$

The left hand side of (7.10) gives the rate of change of angular momentum; since $\vec{\omega}+\dot{\vartheta}$ is of first order in ϵ_1 and the mass is also of first order in ϵ_1 it follows that this quantity is at least of second order in ϵ_1.

We consider the angular momentum with respect to axes $\tilde{x}'$, $\tilde{y}'$, $\tilde{z}'$ through the center of gravity and parallel to the moving $\bar{x}'$, $\bar{y}'$, $\bar{z}'$-system. The angular momentum $\mathfrak{h} = \{H_1, H_2, H_3\}$ has components with respect to these axes given as follows in terms of the components of the angular velocity $-(\vec{\omega}+\dot{\vartheta}) = \{\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3+\omega\}$ and the moments of inertia I_i and products of inertia I_{ij}:

$$-H_1 = I_1\dot{\theta}_1 - I_{13}(\dot{\theta}_3+\omega)$$
$$-H_2 = I_2\dot{\theta}_2$$
$$-H_3 = -I_{13}\dot{\theta}_1 + I_3(\dot{\theta}_3+\omega) \quad (7.21)$$

since only small angular displacements are considered, and the symmetry of the hull leads to $I_{12} = I_{23} = 0$, where we assumed that the mass distribution has the same symmetry as the hull surface. The moments of inertia I_1, I_2 and I_3 are clearly of order ϵ_1:

$$I_i = \epsilon_1 I_{i\,100} \qquad i = 1,2,3 \quad (7.22)$$

as well as the product of inertia I_{13}:

$$I_{13} = \epsilon_1 I_{13\,100} \tag{7.23}$$

Upon using the developments (2.12), (2.22), (7.18) and (7.20) we can write the development of (7.10) in the form:

$$\begin{aligned}
&-n_x\Big[\epsilon_1^2\{I_{1100}\ddot{\Theta}_{1100}-I_{13100}(\ddot{\Theta}_{3100}+\dot{\omega}_{100})\}+\epsilon_1^2\epsilon_2\{I_{1100}\ddot{\Theta}_{1110}-I_{13100}(\ddot{\Theta}_{3110}+\dot{\omega}_{110})\} \\
&\qquad+\epsilon_1^2\epsilon_3\{I_{1100}\ddot{\Theta}_{1101}-I_{13100}(\ddot{\Theta}_{3101}+\dot{\omega}_{101})\}+\epsilon_1^2\epsilon_2\epsilon_3\{I_{1100}\ddot{\Theta}_{1111}-I_{13100}(\ddot{\Theta}_{3111}+\dot{\omega}_{111})\}+\cdots\Big] \\
&-n_y\Big[\epsilon_1^2 I_{2100}\ddot{\Theta}_{2100}+\epsilon_1^2\epsilon_2 I_{2100}\ddot{\Theta}_{2110}+\epsilon_1^2\epsilon_3 I_{2100}\ddot{\Theta}_{2101}+\epsilon_1^2\epsilon_2\epsilon_3 I_{2100}\ddot{\Theta}_{2111}+\cdots\Big] \\
&+n_z\Big[\epsilon_1^2\{I_{13100}\ddot{\Theta}_{1100}-I_{3100}(\ddot{\Theta}_{3100}+\dot{\omega}_{100})\}+\epsilon_1^2\epsilon_2\{I_{13100}\ddot{\Theta}_{1110}-I_{3100}(\ddot{\Theta}_{3110}+\dot{\omega}_{110})\} \\
&\qquad+\epsilon_1^2\epsilon_3\{I_{13100}\ddot{\Theta}_{1101}-I_{3100}(\ddot{\Theta}_{3101}+\dot{\omega}_{101})\}+\epsilon_1^2\epsilon_2\epsilon_3\{I_{13100}\ddot{\Theta}_{1111}-I_{3100}(\ddot{\Theta}_{3111}+\dot{\omega}_{111})\}+\cdots\Big] \\
&= \int_S p(r - z_c n_z)\times n\, dS + n_x\Big[\epsilon_1\bar{t}'_{r100}R_{2100}+\epsilon_1\epsilon_2\bar{t}'_{r110}R_{2110}+\epsilon_1\epsilon_3\bar{t}'_{r101}R_{2101}+\epsilon_1\epsilon_2\epsilon_3\bar{t}'_{r111}R_{2111}+\cdots\Big] \\
&+n_y\Big[\epsilon_1^2\{-\bar{t}'_t I_{200}-(\bar{t}'_{r200}R_{1100}+\bar{t}'_{r100}R_{2100}\Theta_{3100})+(\bar{l}'_{r200}R_{3100}-\bar{l}'_{r100}R_{2100}\Theta_{1100})\} \\
&\qquad+\epsilon_1^2\epsilon_2\{-\bar{t}'_t I_{210}-(\bar{t}'_{r210}R_{1210}+\bar{t}'_{r110}R_{2110}\Theta_{3100}+\bar{t}'_{r100}R_{2100}\Theta_{3110}) \\
&\qquad\qquad+(\bar{l}'_{r210}R_{3210}-\bar{l}'_{r110}R_{2110}\Theta_{1100}-\bar{l}'_{r100}R_{2100}\Theta_{1110})\} \\
&\qquad+\epsilon_1^2\epsilon_3\{-\bar{t}'_t I_{201}-(\bar{t}'_{r201}R_{1201}+\bar{t}'_{r101}R_{2101}\Theta_{3100}+\bar{t}'_{r100}R_{2100}\Theta_{3101}) \\
&\qquad\qquad+(\bar{l}'_{r201}R_{3201}-\bar{l}'_{r101}R_{2101}\Theta_{1100}-\bar{l}'_{r100}R_{2100}\Theta_{1101})\} \\
&\qquad+\epsilon_1^2\epsilon_2\epsilon_3\{-\bar{t}'_t I_{211}-(\bar{t}'_{r211}R_{1211}+\bar{t}'_{r111}R_{2111}\Theta_{3100}+\bar{t}'_{r100}R_{2100}\Theta_{3111}+\bar{t}'_{r110}R_{2110}\Theta_{3101}+\bar{t}'_{r101}R_{2101}\Theta_{3110}) \\
&\qquad\qquad+(\bar{l}'_{r211}R_{3211}-\bar{l}'_{r111}R_{2111}\Theta_{1100}-\bar{l}'_{r100}R_{2100}\Theta_{1111}-\bar{l}'_{r110}R_{2110}\Theta_{1101}-\bar{l}'_{r101}R_{2101}\Theta_{1110})\} \\
&\qquad+\cdots\cdots\cdots\Big] \\
&+n_z\Big[\epsilon_1\bar{l}'_{r100}R_{2100}+\epsilon_1\epsilon_2\bar{l}'_{r110}R_{2110}+\epsilon_1\epsilon_3\bar{l}'_{r101}R_{2101}+\epsilon_1\epsilon_2\epsilon_3\bar{l}'_{r111}R_{2111}+\cdots\Big]
\end{aligned} \tag{7.24}$$

We turn next to the problem of developung the surface intergrals in (7.15) and (7.24). With respect to the $\bar{x}'$, $\bar{y}'$, $\bar{z}'$-system fixed in the ship the equation of the ship's hull has the form

$$\bar{y}' \mp \epsilon_1 h(\bar{x}', \bar{z}') = 0 \qquad \bar{x}', \bar{z}' \text{ on } \bar{A}' \tag{7.25}$$

We begin with the development of the pressure integral in (7.15), but replace the pressure p by the hydrodynamic pressure P (cf. (1.6)):

$$\int_S p\, \mathbf{n}\, dS = -\rho g \int_S z\, \mathbf{n}\, dS + \rho \int_S \mathcal{P}\, \mathbf{n}\, dS \,. \tag{7.26}$$

The surface S over which the integrals are taken is the part of the hull surface submerged in the water. Consider first the term $\rho g \int_S z\, \mathbf{n}\, dS$. If we express this integral in terms of the $\bar{x}'$, $\bar{y}'$, $\bar{z}'$-coordinate system it follows with (2.29):

$$-\rho g \int_S z\, \mathbf{n}\, dS = -\rho g \int_S \Big[\bar{z}' + \epsilon_1 \sum_{j,k=0}^{1} \epsilon_2^j \epsilon_3^k \{z_{1jk} + \bar{x}'\theta_{21jk} - \bar{y}'\theta_{11jk}\} + \cdots\Big] \mathbf{n}\, dS \tag{7.27}$$

On the right hand side the integration is to be carried out over the hull surface submerged in the water with respect to the $\bar{x}'$, $\bar{y}'$, $\bar{z}'$-system fixed in the ship. It is convenient to consider the two sides of the hull separately:

$$\begin{aligned} -\rho g \int_S z\, \mathbf{n}\, dS = &-\rho g \int_{A_1} \Big[\bar{z}' + \epsilon_1 \sum_{j,k=0}^{1} \epsilon_2^j \epsilon_3^k \{z_{1jk} + \bar{x}'\theta_{21jk} - \bar{y}'\theta_{11jk}\} + \cdots\Big] \cdot \\ &\qquad \cdot \big[\epsilon_1 h_{\bar{x}'} \mathbf{n}_{\bar{x}'} - \mathbf{n}_{\bar{y}'} + \epsilon_1 h_{\bar{z}'} \mathbf{n}_{\bar{z}'}\big]\, d\bar{x}'\, d\bar{z}' \\ &-\rho g \int_{A_2} \Big[\bar{z}' + \epsilon_1 \sum_{j,k=0}^{1} \epsilon_2^j \epsilon_3^k \{z_{1jk} + \bar{x}'\theta_{21jk} - \bar{y}'\theta_{11jk}\} + \cdots\Big] \cdot \\ &\qquad \cdot \big[\epsilon_1 h_{\bar{x}'} \mathbf{n}_{\bar{x}'} + \mathbf{n}_{\bar{y}'} + \epsilon_1 h_{\bar{z}'} \mathbf{n}_{\bar{z}'}\big]\, d\bar{x}'\, d\bar{z}' \end{aligned} \tag{7.28}$$

The unit normal $\mathbf{n}$ in the $\bar{x}'$, $\bar{y}'$, $\bar{z}'$-system was expressed through use of (7.25); by A_1 we mean the projection of the immersed part of $\bar{y}' = +\epsilon_1 h(\bar{x}', \bar{z}')$ on the $\bar{x}'$, $\bar{z}'$-plane, by A_2 the projection of $\bar{y}' = -\epsilon_1 h(\bar{x}', \bar{z}')$ on the $\bar{x}'$, $\bar{z}'$-plane.

The free surface displacement of the water differs from the x,y-plane only by a quantity of first order in ϵ_1, since:

$$\begin{aligned} z = \zeta(x,y,t;\epsilon_1,\epsilon_2,\epsilon_3) = &\ \epsilon_1 \zeta_{100}(x,y,t) + \epsilon_1\epsilon_2 \zeta_{110}(x,y,t) \\ &+ \epsilon_1\epsilon_3 \zeta_{101}(x,y,t) + \epsilon_1\epsilon_2\epsilon_3 \zeta_{111}(x,y,t) + \cdots \end{aligned} \tag{7.29}$$

If we substitute in (7.29) the transformation formulas (2.29) we get for the equation of the free surface displacement with respect to the $\bar{x}'$, $\bar{y}'$, $\bar{z}'$-coordinate system the expansion:

$$\begin{aligned}
\bar{z}' = \epsilon_1 [\zeta_{100}(&\bar{x}' \cos\alpha_{000} + \bar{y}' \sin\alpha_{000} + x_{c000}, -\bar{x}' \sin\alpha_{000} + \bar{y}' \cos\alpha_{000} + y_{c000}, t) \\
&- z_{100} - \theta_{2100}\bar{x}' + \theta_{1100}\bar{y}'] \\
+\epsilon_1\epsilon_2 [\zeta_{110}(&\bar{x}' \cos\alpha_{000} + \bar{y}' \sin\alpha_{000} + x_{c000}, -\bar{x}' \sin\alpha_{000} + \bar{y}' \cos\alpha_{000} + y_{c000}, t) \\
&- z_{110} - \theta_{2110}\bar{x}' + \theta_{1110}\bar{y}'] \\
+\epsilon_1\epsilon_3 [\zeta_{101}(&\bar{x}' \cos\alpha_{000} + \bar{y}' \sin\alpha_{000} + x_{c000}, -\bar{x}' \sin\alpha_{000} + \bar{y}' \cos\alpha_{000} + y_{c000}, t) \\
&- z_{101} - \theta_{2101}\bar{x}' + \theta_{1101}\bar{y}'] \\
+\epsilon_1\epsilon_2\epsilon_3 [\zeta_{111}(&\bar{x}' \cos\alpha_{000} + \bar{y}' \sin\alpha_{000} + x_{c000}, -\bar{x}' \sin\alpha_{000} + \bar{y}' \cos\alpha_{000} + y_{c000}, t) \\
&- z_{111} - \theta_{2111}\bar{x}' + \theta_{1111}\bar{y}'] \\
+ \cdots &
\end{aligned} \tag{7.30}$$

From (7.30) follows: The surface displacement of the water differs from the $\bar{x}'$, $\bar{y}'$-plane only by a quantity of first order in ϵ_1. Consider now the part of the hull surface given by (7.25) which lies below the $\bar{x}'$, $\bar{y}'$-plane and call its projection on the $\bar{x}'$, $\bar{z}'$-plane $\bar{A}'$. In Figure 4 this area is shaded. It is now clear that in those portions of A_1 and A_2 which differ from $\bar{A}'$ the quantity $\bar{z}'$ is of order ϵ_1, and in addition, that the areas $A_1 - \bar{A}'$ and $A_2 - \bar{A}'$ are of order ϵ_1. It follows therefore, that (7.28) can be replaced by the equation:

$$\begin{aligned}
-\rho g \int_S z\, n\, dS = &-\rho g \int_{\bar{A}'} [\bar{z}' + \epsilon_1 \sum_{j,k=0}^{1} \epsilon_2^j \epsilon_3^k (z_{1jk} + \bar{x}'\theta_{21jk})] \cdot \\
&\qquad \cdot [\epsilon_1 h_{\bar{x}'} n_{\bar{x}'} - n_{\bar{y}'} + \epsilon_1 h_{\bar{z}'} n_{\bar{z}'}]\, d\bar{x}'\, d\bar{z}' \\
&-\rho g \int_{\bar{A}'} [\bar{z}' + \epsilon_1 \sum_{j,k=0}^{1} \epsilon_2^j \epsilon_3^k (z_{1jk} + \bar{x}'\theta_{21jk})] \cdot \\
&\qquad \cdot [\epsilon_1 h_{\bar{x}'} n_{\bar{x}'} + n_{\bar{y}'} + \epsilon_1 h_{\bar{z}'} n_{\bar{z}'}]\, d\bar{x}'\, d\bar{z}' \\
&+ n_{\bar{x}'} O(\epsilon_1^3) + n_{\bar{y}'} O(\epsilon_1^2) + n_{\bar{z}'} O(\epsilon_1^3) \\
= &-2\rho g \int_{\bar{A}'} [\bar{z}' + \epsilon_1 \sum_{j,k=0}^{1} \epsilon_2^j \epsilon_3^k (z_{1jk} + \bar{x}'\theta_{21jk})] \cdot \\
&\qquad \cdot [\epsilon_1 h_{\bar{x}'} n_{\bar{x}'} + \epsilon_1 h_{\bar{z}'} n_{\bar{z}'}]\, d\bar{x}'\, d\bar{z}' \\
&+ n_{\bar{x}'} O(\epsilon_1^3) + n_{\bar{y}'} O(\epsilon_1^2) + n_{\bar{z}'} O(\epsilon_1^3) \quad .
\end{aligned} \tag{7.31}$$

The last integral can be given a different form by means of the following relations. In fact, since $h = 0$ at all boundary points of $\bar{A}'$ for which $z' < 0$ (from the symmetry and continuity of the hull surface) the following formulas can be derived through integration by parts:

$$\int_{\bar{A}'} h_{\bar{x}'}\, d\bar{x}' d\bar{z}' = 0 \qquad \int_{\bar{A}'} h_{\bar{z}'}\, d\bar{x}' d\bar{z}' = \int_{\bar{L}'} h\, d\bar{x}'$$

$$\int_{\bar{A}'} \bar{z}' h_{\bar{x}'}\, d\bar{x}' d\bar{z}' = 0 \qquad \int_{\bar{A}'} \bar{z}' h_{\bar{z}'}\, d\bar{x}' d\bar{z}' = -\int_{\bar{A}'} h\, d\bar{x}' d\bar{z}'$$

$$\int_{\bar{A}'} \bar{x}' h_{\bar{x}'}\, d\bar{x}' d\bar{z}' = -\int_{\bar{A}'} h\, d\bar{x}' d\bar{z}' \qquad \int_{\bar{A}'} \bar{x}' h_{\bar{z}'}\, d\bar{x}' d\bar{z}' = \int_{\bar{L}'} \bar{x}' h\, d\bar{x}' \tag{7.32}$$

where the line integrals are taken over the part $\bar{L}'$ of the boundary of $\bar{A}'$ which lies in the $\bar{x}'$-axes. Insertion of (7.32) in (7.31) yields:

$$\begin{aligned}-\rho g \int_S z\, n\, dS = {} & n_{\bar{x}'}\Big[2\rho g \int_{\bar{A}'} h\, d\bar{x}' d\bar{z}' \cdot \epsilon_1^2 \sum_{j,k=0}^{1} \epsilon_2^j \epsilon_3^k \theta_{21jk} + O(\epsilon_1^3)\Big] \\ & + n_{\bar{y}'}\Big[O(\epsilon_1^2)\Big] + n_{\bar{z}'}\Big[2\rho g \int_{\bar{A}'} h\, d\bar{x}' d\bar{z}' \cdot \epsilon_1 \\ & - 2\rho g \epsilon_1^2 \sum_{j,k=0}^{1} \epsilon_2^j \epsilon_3^k \Big\{\int_{\bar{L}'} (z_{1jk} + \bar{x}'\theta_{21jk}) h\, d\bar{x}'\Big\} + O(\epsilon_1^3)\Big]\end{aligned} \tag{7.33}$$

If we now transform to the $\bar{x}$, $\bar{y}$, $\bar{z}$-coordinate system we find on account of (2.26):

$$\begin{aligned}-2\rho g \int_S z\, n\, dS = {} & n_{\bar{x}}\Big[O(\epsilon_1^3)\Big] \\ & + n_{\bar{y}}\Big[O(\epsilon_1^2)\Big] \\ & + n_{\bar{z}}\Big[\epsilon_1\, 2\rho g \int_{\bar{A}'} h\, d\bar{x}\, d\bar{z} - 2\rho g \epsilon_1^2 \sum_{j,k=0}^{1} \epsilon_2^j \epsilon_3^k \int_{\bar{L}'} (z_{1jk} + \bar{x}\theta_{21jk}) h\, d\bar{x} + O(\epsilon_1^3)\Big]\end{aligned} \tag{7.34}$$

Here we made use of the fact that the development of all quantities takes place about the rest position of equilibrium of the hull; therefore, in this position the two moving systems $\bar{x}'$, $\bar{y}'$, $\bar{z}'$ and $\bar{x}$, $\bar{y}$, $\bar{z}$ coincide, and hence we could replace the integration variables $\bar{x}'$, $\bar{z}'$ by $\bar{x}$, $\bar{z}$ in the integrals of (7.34). Next we calculate the development for the second integral on the right hand side of (7.26). Here the discussion takes the same general course as the one just completed. Hence we give the steps in summary fashion only:

$$\rho\int_S p\, w\, dS = \rho\int_S \bar{p}\, w\, dS$$

$$= \rho\int_{A_1}\Big[\epsilon_1\sum_{j,k=0}^{1}\epsilon_2^j\epsilon_3^k\,\bar{p}_{1jk}(\bar{x}',0^+,\bar{z}',t)\Big]\Big[\epsilon_1 h_{\bar{x}'} n_{\bar{x}'} - n_{\bar{y}'} + \epsilon_1 h_{\bar{z}'} n_{\bar{z}'}\Big]d\bar{x}'d\bar{z}'$$

$$+\rho\int_{\bar{A}_2}\Big[\epsilon_1\sum_{j,k=0}^{1}\epsilon_2^j\epsilon_3^k\,\bar{p}_{1jk}(\bar{x}',0^-,\bar{z}',t)\Big]\Big[\epsilon_1 h_{\bar{x}'} n_{\bar{x}'} + n_{\bar{y}'} + \epsilon_1 h_{\bar{z}'} n_{\bar{z}'}\Big]d\bar{x}'d\bar{z}'$$

$$+ n_{\bar{x}'}[O(\epsilon_1^3)] + n_{\bar{y}'}[O(\epsilon_1^2)] + n_{\bar{z}'}[O(\epsilon_1^3)]$$

$$= n_{\bar{x}'}\Big[\rho\,\epsilon_1^2\sum_{j,k=0}^{1}\epsilon_2^j\epsilon_3^k\int_{\bar{A}'}(\bar{p}^+_{1jk}+\bar{p}^-_{1jk})h_{\bar{x}'}\,d\bar{x}'d\bar{z}' + O(\epsilon_1^3)\Big]$$

$$-n_{\bar{y}'}\Big[\rho\,\epsilon_1\sum_{j,k=0}^{1}\epsilon_2^j\epsilon_3^k\int_{\bar{A}'}(\bar{p}^+_{1jk}-\bar{p}^-_{1jk})\,d\bar{x}'d\bar{z}' + O(\epsilon_1^2)\Big]$$

$$+n_{\bar{z}'}\Big[\rho\,\epsilon_1^2\sum_{j,k=0}^{1}\epsilon_2^j\epsilon_3^k\int_{\bar{A}'}(\bar{p}^+_{1jk}+\bar{p}^-_{1jk})h_{\bar{z}'}\,d\bar{x}'d\bar{z}' + O(\epsilon_1^3)\Big]$$

$$= n_{\bar{x}}\Big[\rho\,\epsilon_1^2\Big\{\int_{\bar{A}'}\big[(h_{\bar{x}'}-\theta_{3100})\bar{p}^+_{100} + (h_{\bar{x}'}+\theta_{3100})\bar{p}^-_{100}\big]d\bar{x}d\bar{z}\Big\}$$

$$+\rho\,\epsilon_1^2\epsilon_2\Big\{\int_{\bar{A}'}\big[(h_{\bar{x}'}-\theta_{3100})\bar{p}^+_{110} + (h_{\bar{x}'}+\theta_{3100})\bar{p}^-_{110} - \theta_{3110}(\bar{p}^+_{100}-\bar{p}^-_{100})\big]d\bar{x}d\bar{z}\Big\}$$

$$+\rho\,\epsilon_1^2\epsilon_3\Big\{\int_{\bar{A}'}\big[(h_{\bar{x}'}-\theta_{3100})\bar{p}^+_{101} + (h_{\bar{x}'}+\theta_{3100})\bar{p}^-_{101} - \theta_{3101}(\bar{p}^+_{100}-\bar{p}^-_{100})\big]d\bar{x}d\bar{z}\Big\}$$

$$+ g\varepsilon_1^2\varepsilon_2\varepsilon_3\Big\{\int_{\bar{A}'}[(h_{\bar{x}'} - \Theta_{3100})\bar{p}^+_{111} + (h_{\bar{x}'} + \Theta_{3100})\bar{p}^-_{111} - \Theta_{3111}(\bar{p}^+_{100} - \bar{p}^-_{100})$$

$$- \Theta_{3110}(\bar{p}^+_{101} - \bar{p}^-_{101}) - \Theta_{3101}(\bar{p}^+_{110} - \bar{p}^-_{110})]\,d\bar{x}\,d\bar{z}\Big\}$$

$$+ \quad . \quad . \quad . \quad \Big]$$

$$-n_{\bar{y}}\Big[g\varepsilon_1\int_{\bar{A}'}(\bar{p}^+_{100} - \bar{p}^-_{100})\,d\bar{x}\,d\bar{z} + g\varepsilon_1\varepsilon_2\int_{\bar{A}'}(\bar{p}^+_{101} - \bar{p}^-_{101})\,d\bar{x}\,d\bar{z}$$

$$+ g\varepsilon_1\varepsilon_3\int_{\bar{A}'}(\bar{p}^+_{101} - \bar{p}^-_{101})\,d\bar{x}\,d\bar{z} + g\,\varepsilon_1\varepsilon_2\varepsilon_3\int_{\bar{A}'}(\bar{p}^+_{111} - \bar{p}^-_{111})\,d\bar{x}\,d\bar{z} + \cdots\Big]$$

$$+n_{\bar{z}}\Big[g\varepsilon_1^2\Big\{\int_{\bar{A}'}[(h_{\bar{z}'} + \Theta_{1100})\bar{p}^+_{100} + (h_{\bar{z}'} - \Theta_{1100})\bar{p}^-_{100}]\,d\bar{x}\,d\bar{z}\Big\}$$

$$+g\varepsilon_1^2\varepsilon_2\Big\{\int_{\bar{A}'}[(h_{\bar{z}'} + \Theta_{1100})\bar{p}^+_{110} + (h_{\bar{z}'} - \Theta_{1100})\bar{p}^-_{110} + \Theta_{1110}(\bar{p}^+_{100} - \bar{p}^-_{100})]\,d\bar{x}\,d\bar{z}\Big\}$$

$$+g\varepsilon_1^2\varepsilon_3\Big\{\int_{\bar{A}'}[(h_{\bar{z}'} + \Theta_{1100})\bar{p}^+_{101} + (h_{\bar{z}'} - \Theta_{1100})\bar{p}^-_{101} + \Theta_{1101}(\bar{p}^+_{100} - \bar{p}^-_{100})]\,d\bar{x}\,d\bar{z}\Big\}$$

$$+g\varepsilon_1^2\varepsilon_2\varepsilon_3\Big\{\int_{\bar{A}'}[(h_{\bar{z}'} + \Theta_{1100})\bar{p}^+_{111} + (h_{\bar{z}'} - \Theta_{1100})\bar{p}^-_{111} + \Theta_{1111}(\bar{p}^+_{100} - \bar{p}^-_{100})$$

$$+ \Theta_{1110}(\bar{p}^+_{101} - \bar{p}^-_{101}) + \Theta_{1101}(\bar{p}^+_{110} - \bar{p}^-_{110})]\,d\bar{x}\,d\bar{z}\Big\}$$

$$+ \quad . \quad . \quad . \quad . \quad \Big] \tag{7.35}$$

where we signified with $\bar{P}^+_{1jk}$ and $\bar{P}^-_{1jk}$ the limiting values of the hydrodynamical pressures $\bar{P}_{1jk}$ $(j, k = 0,1)$ on the two sides $\bar{A}'^+$ and $\bar{A}'^-$ of the disk $\bar{A}'$. This is necessary here since the hydrodynamical pressures $\bar{P}_{1jk}$ $(j,k = 0,1)$ are in general not continuous across the disk.

Substitution of the developments (7.34) and (7.35) in (7.15) gives the following scalar equations:

$$\dot{s}_{000} = 0 \tag{7.36}$$

$$M_{100}\,\dot{s}_{100} = \rho \int_{A'} (\bar{p}^{+}_{100} + \bar{p}^{-}_{100})\, h_{\bar{x}'}\, d\bar{x}\, d\bar{z} + T_{200} + R_{1200} \tag{7.37}$$

$$M_{100}\,\dot{s}_{110} = \rho \int_{A'} (\bar{p}^{+}_{110} + \bar{p}^{-}_{110})\, h_{\bar{x}'}\, d\bar{x}\, d\bar{z} + T_{210} + R_{1210} \tag{7.38}$$

$$M_{100}\,\dot{s}_{101} = \rho \int_{A'} (\bar{p}^{+}_{101} + \bar{p}^{-}_{101})\, h_{\bar{x}'}\, d\bar{x}\, d\bar{z} + T_{201} + R_{1201} \tag{7.39}$$

$$M_{100}\,\dot{s}_{111} = \rho \int_{A'} (\bar{p}^{+}_{111} + \bar{p}^{-}_{111})\, h_{\bar{x}'}\, d\bar{x}\, d\bar{z} + T_{211} + R_{1211} \tag{7.40}$$

. . . .

$$\rho \int_{A'} (\bar{p}^{+}_{100} - \bar{p}^{-}_{100})\, d\bar{x}\, d\bar{z} = R_{2100} \tag{7.41}$$

$$\rho \int_{A'} (\bar{p}^{+}_{110} - \bar{p}^{-}_{110})\, d\bar{x}\, d\bar{z} = R_{2110} \tag{7.42}$$

$$\rho \int_{A'} (\bar{p}^{+}_{101} - \bar{p}^{-}_{101})\, d\bar{x}\, d\bar{z} = R_{2101} \tag{7.43}$$

$$\rho \int_{A'} (\bar{p}^{+}_{111} - \bar{p}^{-}_{111})\, d\bar{x}\, d\bar{z} = R_{2111} \tag{7.44}$$

. . . . . .

$$M_{100}\, g - 2\rho g \int_{A'} h\, d\bar{x}\, d\bar{z} = 0 \tag{7.45}$$

$$M_{100}\,\ddot{z}_{100} = -2\rho g \int_{L'} (z_{100} + \bar{x}\,\theta_{2100})\, h\, d\bar{x} + \int_{A'} (\bar{p}^{+}_{100} + \bar{p}^{-}_{100})\, h_{\bar{x}'}\, d\bar{x}\, d\bar{z} + R_{3200} \tag{7.46}$$

$$M_{100}\,\ddot{z}_{110} = -2\rho g \int_{\bar L'} (z_{110} + \bar x\,\Theta_{2110})\,h\,d\bar x + \int_{\bar A'} (\bar p^+_{110} + \bar p^-_{110})\,h_{\bar z'}\,d\bar x\,d\bar z + R_{3210} \tag{7.47}$$

$$M_{100}\,\ddot{z}_{101} = -2\rho g \int_{\bar L'} (z_{101} + \bar x\,\Theta_{2101})\,h\,d\bar x + \int_{\bar A'} (\bar p^+_{101} + \bar p^-_{101})\,h_{\bar z'}\,d\bar x\,d\bar z + R_{3201} \tag{7.48}$$

$$M_{100}\,\ddot{z}_{111} = -2\rho g \int_{\bar L'} (z_{111} + \bar x\,\Theta_{2111})\,h\,d\bar x + \int_{\bar A'} (\bar p^+_{111} + \bar p^-_{111})\,h_{\bar z'}\,d\bar x\,d\bar z + R_{3211} \tag{7.49}$$

. . . . . .

In (7.37) - (7.40) and (7.46) - (7.49) we have already used the relations (4.41) - (4.44).

The interpretation and significance of this equation is analogously to that of the corresponding equations of Peters and Stoker(9) and will not be discussed here. Next we turn to the development of the surface integral in the angular momentum equation (7.24). Again the calculations proceed in the same fashion as above and hence we can cut them short. We first introduce the hydrodynamical pressure P instead of p by use of (1.6). Then we have:

$$\begin{aligned}\int_S p(\bar r - \bar z_c n_{\bar z}) \times n\,dS &= -\rho g \int_S z(\bar r - \bar z_c n_{\bar z}) \times n\,dS \\ &\quad + \rho \int_S P(\bar r - \bar z_c n_{\bar z}) \times n\,dS \\ &= -\rho g \int_S \bar z(\bar r - \bar z_c n_{\bar z}) \times n\,dS + \rho \int_S \bar P(\bar r - \bar z_c n_{\bar z}) \times n\,dS\end{aligned} \tag{7.50}$$

Consider the first term $-\rho g \int_S \bar z(\bar r - \bar z_c n_{\bar z}) \times n\,dS$.
Inserting (2.25), (2.27), (7.7) and (7.25) it follows that:

$$-\rho g \int_S \bar z(\bar r - \bar z_c n_{\bar z}) \times n\,dS$$

$$= -\rho g \int_{A_1} \Big[\bar z' + \epsilon_1 \sum_{j,k=0}^{1} \epsilon_2^j \epsilon_3^k (z_{1jk} + \Theta_{21jk}\,\bar x' - \Theta_{11jk}\,\bar y') + \dots\Big] \cdot$$

$$\cdot \big[\bar x' n_{\bar x'} + \bar y' n_{\bar y'} + (\bar z' - \bar z_c') n_{\bar z'} + \dots\big] \times \big[\epsilon_1 h_{\bar x'} n_{\bar x'} - n_{\bar y'} + \epsilon_1 h_{\bar z'} n_{\bar z'}\big]\,d\bar x'\,d\bar z'$$

$$-\rho g\int_{A_2}\Big[\bar z' + \varepsilon_1\sum_{j,k=0}^{1}\varepsilon_2^j\varepsilon_3^k(z_{1jk}+\Theta_{21jk}\bar x' - \Theta_{11jk}\bar y') + \cdots\Big]\cdot$$

$$\cdot\big[\bar x' n_{\bar x'} + \bar y' n_{\bar y'} + (\bar z' - \bar z_c')n_{\bar z'} + \cdots\big] \times \big[\varepsilon_1 h_{\bar x'} n_{\bar x'} + n_{\bar y'} + \varepsilon_1 h_{\bar z'} n_{\bar z'}\big] d\bar x' d\bar z'$$

$$= -2\rho g\int_{\bar A'}\Big[\bar z' + \varepsilon_1\sum_{j,k=0}^{1}\varepsilon_2^j\varepsilon_3^k(z_{1jk}+\Theta_{21jk}\bar x' - \Theta_{11jk}\bar y')\Big]\cdot$$

$$\cdot\big[\bar x' n_{\bar x'} + \bar y' n_{\bar y'} + (\bar z' - \bar z_c')n_{\bar z'} + \cdots\big]\times\big[\varepsilon_1 h_{\bar z'} n_{\bar x'} + \varepsilon_1 h_{\bar z'} n_{\bar z'}\big] d\bar x' d\bar z'$$

$$+ n_{\bar x'}[O(\varepsilon_1^2)] + n_{\bar y'}[O(\varepsilon_1^3)] + n_{\bar z'}[O(\varepsilon_1^2)]$$

$$= -2\rho g\int_{\bar A'}\Big[\bar z' + \varepsilon_1\sum_{j,k=0}^{1}\varepsilon_2^j\varepsilon_3^k(z_{1jk} + \bar x'\Theta_{21jk})\Big]\cdot$$

$$\cdot\big[\bar x' n_{\bar x'} + (\bar z' - \bar z_c')n_{\bar z'}\big]\times\big[\varepsilon_1 h_{\bar x'} n_{\bar x'} + \varepsilon_1 h_{\bar z'} n_{\bar z'}\big] d\bar x' d\bar z'$$

$$+ n_{\bar x'}[O(\varepsilon_1^2)] + n_{\bar y'}[O(\varepsilon_1^3)] + n_{\bar z'}[O(\varepsilon_1^2)]$$

$$= n_{\bar x'}[O(\varepsilon_1^2)]$$

$$+ n_{\bar y'}\Big[\varepsilon_1 2\rho g\int_{\bar A'}\bar z'\{\bar x' h_{\bar z'} - (\bar z' - \bar z_c')h_{\bar x'}\} d\bar x' d\bar z'$$

$$+ 2\rho g\,\varepsilon_1^2\sum_{j,k=0}^{1}\varepsilon_2^j\varepsilon_3^k\int_{\bar A'}(z_{1jk} + \bar x'\Theta_{21jk})\{\bar x' h_{\bar z'} - (\bar z' - \bar z_c')h_{\bar x'}\} d\bar x' d\bar z' + O(\varepsilon_1^3)\Big]$$

$$+ n_{\bar z'}[O(\varepsilon_1^2)]$$

$$= n_x[O(\varepsilon_1^2)] + n_y\Big[-2\rho g\int_{\bar A'}\bar x h\, d\bar x\, d\bar z\cdot\varepsilon_1$$

$$+ 2\rho g\,\varepsilon_1^2\sum_{j,k=0}^{1}\varepsilon_2^j\varepsilon_3^k\Big\{\int_{\bar L'}(z_{1jk} + \bar x\Theta_{21jk})\bar x h\, d\bar x$$

$$+ \Theta_{21jk}\int_{\bar A'}(\bar z - \bar z_c')h\, d\bar x\, d\bar z\Big\} + \cdots\Big] + n_z[O(\varepsilon_1^2)] \quad (7.51)$$

where we have used the formulas

$$\int_{\bar{A}'} \bar{x}'\bar{z}' h_{\bar{z}'}\, d\bar{x}'\, d\bar{z}' = -\int_{\bar{A}'} \bar{x}' h\, d\bar{x}'\, d\bar{z}' \; ; \qquad \int_{\bar{A}'} \bar{x}'(\bar{z}'-\bar{z}'_c) h_{\bar{z}'}\, d\bar{x}'\, d\bar{z}' = 0 \; ;$$

$$\int_{\bar{A}'} (\bar{z}'-\bar{z}'_c)\bar{x}' h_{\bar{x}'}\, d\bar{x}'\, d\bar{z}' = -\int_{\bar{A}'} (\bar{z}'-\bar{z}'_c) h\, d\bar{x}'\, d\bar{z}' \; , \qquad (7.52)$$

which can be derived through integration by parts. In an analogous manner we find for the second integral on the right hand side of (7.50):

$$\rho \int_S \bar{p}\,(\bar{r} - \xi n_\xi) \times n\, dS$$

$$= \rho \int_{A_1} \Big[\varepsilon_1 \sum_{j,k=0}^{1} \varepsilon_2^j \varepsilon_3^k\, \bar{p}_{1jk}(\bar{x}',0^+,\bar{z}',t)\Big]\Big[\bar{x}' n_{\bar{x}'} + \bar{y}' n_{\bar{y}'} + (\bar{z}'-\bar{z}'_c) n_{\bar{z}'}\Big] \times \Big[\varepsilon_1 h_{\bar{x}'} n_{\bar{x}'} - n_{\bar{y}'} + \varepsilon_1 h_{\bar{z}'} n_{\bar{z}'}\Big]\, d\bar{x}'\, d\bar{z}'$$

$$+ \rho \int_{A_2} \Big[\varepsilon_1 \sum_{j,k=0}^{1} \varepsilon_2^j \varepsilon_3^k\, \bar{p}_{1jk}(\bar{x}',0^-,\bar{z}',t)\Big]\Big[\bar{x}' n_{\bar{x}'} + \bar{y}' n_{\bar{y}'} + (\bar{z}'-\bar{z}'_c) n_{\bar{z}'}\Big] \times \Big[\varepsilon_1 h_{\bar{x}'} n_{\bar{x}'} + n_{\bar{y}'} + \varepsilon_1 h_{\bar{z}'} n_{\bar{z}'}\Big]\, d\bar{x}'\, d\bar{z}'$$

$$+ n_{\bar{x}'}[O(\varepsilon_1^2)] + n_{\bar{y}'}[O(\varepsilon_1^3)] + n_{\bar{z}'}[O(\varepsilon_1^2)]$$

$$= \rho \int_{\bar{A}'} \Big[\varepsilon_1 \sum_{j,k=0}^{1} \varepsilon_2^j \varepsilon_3^k\, \bar{p}_{1jk}(\bar{x}',0^+,\bar{z}',t)\Big]\Big[\bar{x}' n_{\bar{x}'} + (\bar{z}'-\bar{z}'_c) n_{\bar{z}'}\Big] \times \Big[\varepsilon_1 h_{\bar{x}'} n_{\bar{x}'} - n_{\bar{y}'} + \varepsilon_1 h_{\bar{z}'} n_{\bar{z}'}\Big]\, d\bar{x}'\, d\bar{z}'$$

$$+ \rho \int_{\bar{A}'} \Big[\varepsilon_1 \sum_{j,k=0}^{1} \varepsilon_2^j \varepsilon_3^k\, \bar{p}_{1jk}(\bar{x}',0^-,\bar{z}',t)\Big]\Big[\bar{x}' n_{\bar{x}'} + (\bar{z}'-\bar{z}'_c) n_{\bar{z}'}\Big] \times \Big[\varepsilon_1 h_{\bar{x}'} n_{\bar{x}'} + n_{\bar{y}'} + \varepsilon_1 h_{\bar{z}'} n_{\bar{z}'}\Big]\, d\bar{x}'\, d\bar{z}'$$

$$+ n_{\bar{x}'}[O(\varepsilon_1^2)] + n_{\bar{y}'}[O(\varepsilon_1^3)] + n_{\bar{z}'}[O(\varepsilon_1^2)]$$

$$= n_{\bar{x}'}\Big[\rho\varepsilon_1 \sum_{j,k=0}^{1} \varepsilon_2^j \varepsilon_3^k \int_{\bar{A}'} (\bar{p}^+_{1jk} - \bar{p}^-_{1jk})(\bar{z}'-\bar{z}'_c)\, d\bar{x}'\, d\bar{z}' + O(\varepsilon_1^2)\Big]$$

$$+ n_{\bar{y}'}\Big[-\rho\varepsilon_1^2 \sum_{j,k=0}^{1} \varepsilon_2^j \varepsilon_3^k \int_{\bar{A}'} (\bar{p}^+_{1jk} + \bar{p}^-_{1jk})\big[\bar{x}' h_{\bar{z}'} - (\bar{z}'-\bar{z}'_c) h_{\bar{x}'}\big]\, d\bar{x}'\, d\bar{z}' + O(\varepsilon_1^3)\Big]$$

$$+ n_{\bar{z}'}\Big[-\rho\varepsilon_1 \sum_{j,k=0}^{1} \varepsilon_2^j \varepsilon_3^k \int (\bar{p}^+_{1jk} - \bar{p}^-_{1jk})\,\bar{x}'\, d\bar{x}'\, d\bar{z}' + O(\varepsilon_1^2)\Big]$$

$$
\begin{aligned}
= m_{\bar x}\Big[& g\epsilon_1 \sum_{j,k=0}^{1} \epsilon_2^j \epsilon_3^k \int_{\bar A'} (\bar\rho^+_{1jk} - \bar\rho^-_{1jk})(\bar z - \bar z'_c)\, d\bar x\, d\bar z + O(\epsilon_1^2) \Big] \\
+ m_{\bar y}\Big[& -\epsilon_1^2 g \Big\{ \int_{\bar A'} (\bar\rho^+_{100} - \bar\rho^-_{100})[\bar x\,\Theta_{1100} + (\bar z - \bar z'_c)\,\Theta_{3100}]\, d\bar x\, d\bar z \\
& \qquad + \int_{\bar A'} (\bar\rho^+_{100} + \bar\rho^-_{100})[\bar x\, h_{\bar z'} - (\bar z - \bar z'_c)\, h_{\bar x'}]\, d\bar x\, d\bar z \Big\} \\
& -\epsilon_1^2\epsilon_2 g \Big\{ \int_{\bar A'} \big[(\bar\rho^+_{110} - \bar\rho^-_{110})[\bar x\,\Theta_{1100} + (\bar z - \bar z'_c)\,\Theta_{3100}] + (\bar\rho^+_{100} - \bar\rho^-_{100})[\bar x\,\Theta_{1110} + (\bar z - \bar z'_c)\,\Theta_{3110}]\big]\, d\bar x\, d\bar z \\
& \qquad + \int_{\bar A'} (\bar\rho^+_{110} + \bar\rho^-_{110})[\bar x\, h_{\bar z'} - (\bar z - \bar z'_c)\, h_{\bar x'}]\, d\bar x\, d\bar z \Big\} \\
& -\epsilon_1^2\epsilon_3 g \Big\{ \int_{\bar A'} \big[(\bar\rho^+_{101} - \bar\rho^-_{101})[\bar x\,\Theta_{1100} + (\bar z - \bar z'_c)\,\Theta_{3100}] + (\bar\rho^+_{100} - \bar\rho^-_{100})[\bar x\,\Theta_{1101} + (\bar z - \bar z'_c)\,\Theta_{3101}]\big]\, d\bar x\, d\bar z \\
& \qquad + \int_{\bar A'} (\bar\rho^+_{101} + \bar\rho^-_{101})[\bar x\, h_{\bar z'} - (\bar z - \bar z'_c)\, h_{\bar x'}]\, d\bar x\, d\bar z \Big\} \\
& -\epsilon_1^2\epsilon_2\epsilon_3 g \Big\{ \int_{\bar A'} \big[(\bar\rho^+_{111} - \bar\rho^-_{111})[\bar x\,\Theta_{1100} + (\bar z - \bar z'_c)\,\Theta_{3100}] + (\bar\rho^+_{100} - \bar\rho^-_{100})[\bar x\,\Theta_{1111} + (\bar z - \bar z'_c)\,\Theta_{3111}] \\
& \qquad + (\bar\rho^+_{110} - \bar\rho^-_{110})[\bar x\,\Theta_{1101} + (\bar z - \bar z'_c)\,\Theta_{3101}] + (\bar\rho^+_{101} - \bar\rho^-_{101})[\bar x\,\Theta_{1110} + (\bar z - \bar z'_c)\,\Theta_{3110}]\big]\, d\bar x\, d\bar z \\
& \qquad + \int_{\bar A'} (\bar\rho^+_{111} + \bar\rho^-_{111})[\bar x\, h_{\bar z'} - (\bar z - \bar z'_c)\, h_{\bar x'}]\, d\bar x\, d\bar z \Big\} \\
& + \quad \cdot \quad \cdot \quad \cdot \Big] \\
+ m_{\bar z}\Big[& -g\epsilon_1 \sum_{j,k=0}^{1} \epsilon_2^j \epsilon_3^k \int_{\bar A'} (\bar\rho^+_{1jk} - \bar\rho^-_{1jk})\, \bar x\, d\bar x\, d\bar z + O(\epsilon_1^2) \Big] \qquad (7.53)
\end{aligned}
$$

Insertion of the developments (7.51) and (7.53) in (7.24) yields the following scalar equations:

$$\varrho \int_{\bar{A}'} (\bar{p}^{+}_{100} - \bar{p}^{-}_{100})(\bar{z} - \bar{z}'_c)\, d\bar{x}\, d\bar{z} + \bar{t}'_{\tau 100} R_{2100} = 0 \tag{7.54}$$

$$\varrho \int_{\bar{A}'} (\bar{p}^{+}_{110} - \bar{p}^{-}_{110})(\bar{z} - \bar{z}'_c)\, d\bar{x}\, d\bar{z} + \bar{t}'_{\tau 110} R_{2110} = 0 \tag{7.55}$$

$$\varrho \int_{\bar{A}'} (\bar{p}^{+}_{101} - \bar{p}^{-}_{101})(\bar{z} - \bar{z}'_c)\, d\bar{x}\, d\bar{z} + \bar{t}'_{\tau 101} R_{2101} = 0 \tag{7.56}$$

$$\varrho \int_{\bar{A}'} (\bar{p}^{+}_{111} - \bar{p}^{-}_{111})(\bar{z} - \bar{z}'_c)\, d\bar{x}\, d\bar{z} + \bar{t}'_{\tau 111} R_{2111} = 0 \tag{7.57}$$

. . . . . .

$$\int_{\bar{A}'} \bar{x}\, h\, d\bar{x}\, d\bar{z} = 0 \tag{7.58}$$

$$\begin{aligned} I_{2100}\ddot{\Theta}_{2100} = & -2\varrho g \left\{ \int_{\bar{l}'} (\bar{z}_{100} + \bar{x}\Theta_{2100})\bar{x} h\, d\bar{x} + \Theta_{2100} \int_{\bar{A}'} (\bar{z} - \bar{z}'_c) h\, d\bar{x}\, d\bar{z} \right\} \\ & + \varrho \int_{\bar{A}'} (\bar{p}^{+}_{100} + \bar{p}^{-}_{100})[\bar{x} h_{\bar{z}'} - (\bar{z} - \bar{z}'_c) h_{\bar{x}'}]\, d\bar{x}\, d\bar{z} + \bar{t}'_t T_{200} + \bar{t}'_{\tau 200} R_{1200} - \bar{\ell}'_{\tau 200} R_{3200} \end{aligned} \tag{7.59}$$

$$\begin{aligned} I_{2100}\ddot{\Theta}_{2110} = & -2\varrho g \left\{ \int_{\bar{l}'} (\bar{z}_{110} + \bar{x}\Theta_{2110})\bar{x} h\, d\bar{x} + \Theta_{2110} \int_{\bar{A}'} (\bar{z} - \bar{z}'_c) h\, d\bar{x}\, d\bar{z} \right\} \\ & + \varrho \int_{\bar{A}'} (\bar{p}^{+}_{110} + \bar{p}^{-}_{110})[\bar{x} h_{\bar{z}'} - (\bar{z} - \bar{z}'_c) h_{\bar{x}'}]\, d\bar{x}\, d\bar{z} + \bar{t}'_t T_{210} + \bar{t}'_{\tau 210} R_{1210} - \bar{\ell}'_{\tau 210} R_{3210} \end{aligned} \tag{7.60}$$

$$I_{2100}\ddot{\Theta}_{2101} = -2\rho g\left\{\int_{\bar{L}'}(z_{101}+\bar{x}\Theta_{2101})\bar{x}h\,d\bar{x} + \Theta_{2101}\int_{\bar{A}'}(\bar{z}-\bar{z}_c')h\,d\bar{x}\,d\bar{z}\right\}$$

$$+\rho\int_{\bar{A}'}(\bar{p}^{+}_{101}+\bar{p}^{-}_{101})[\bar{x}h_{\bar{z}'}-(\bar{z}-\bar{z}_c')h_{\bar{x}'}]\,d\bar{x}\,d\bar{z}+\bar{t}'_t T_{201}+\bar{t}'_{t201}R_{1201}-\bar{l}'_{t201}R_{3201} \qquad (7.61)$$

$$I_{2100}\ddot{\Theta}_{2111} = -2\rho g\left\{\int_{\bar{L}'}(z_{111}+\bar{x}\Theta_{2111})\bar{x}h\,d\bar{x} + \Theta_{2111}\int_{\bar{A}'}(\bar{z}-\bar{z}_c')h\,d\bar{x}\,d\bar{z}\right\}$$

$$+\rho\int_{\bar{A}'}(\bar{p}^{+}_{111}+\bar{p}^{-}_{111})[\bar{x}h_{\bar{z}'}-(\bar{z}-\bar{z}_c')h_{\bar{x}'}]\,d\bar{x}\,d\bar{z}+\bar{t}'_t T_{211}+\bar{t}'_{t211}R_{1211}-\bar{l}'_{t211}R_{3211}$$

$$(7.62)$$

. . . . . .

$$\rho\int_{\bar{A}'}(\bar{p}^{+}_{100}-\bar{p}^{-}_{100})\bar{x}\,d\bar{x}\,d\bar{z} + \bar{l}'_{t100}R_{2100} = 0 \qquad (7.63)$$

$$\rho\int_{\bar{A}'}(\bar{p}^{+}_{110}-\bar{p}^{-}_{110})\bar{x}\,d\bar{x}\,d\bar{z} + \bar{l}'_{t110}R'_{2110} = 0 \qquad (7.64)$$

$$\rho\int_{\bar{A}'}(\bar{p}^{+}_{101}-\bar{p}^{-}_{101})\bar{x}\,d\bar{x}\,d\bar{z} + \bar{l}'_{t101}R_{2101} = 0 \qquad (7.65)$$

$$\rho\int_{\bar{A}'}(\bar{p}^{+}_{111}-\bar{p}^{-}_{111})\bar{x}\,d\bar{x}\,d\bar{z} + \bar{l}'_{t111}R_{2111} = 0 \qquad (7.66)$$

. . . . .

These equations together with the integral equations of section 7 and initial conditions serve to determine the motion of our ship under the action of a constant propeller thrust and rudder force.

Consider for instance one of our boundary value problems which is steady-state when referred to the moving $\bar{x}$, $\bar{y}$, $\bar{z}$-corrdinate system. Here the equations

$$\varrho \int_{\bar{A}'} (\bar{p}^{+}_{1jk} - \bar{p}^{-}_{1jk})\, d\bar{x}\, d\bar{z} = R_{21jk} \qquad (7.67)$$

$$-\varrho \int_{\bar{A}'} (\bar{p}^{+}_{1jk} - \bar{p}^{-}_{1jk})(\bar{z} - \bar{z}_c')\, d\bar{x}\, d\bar{z} = \bar{t}'_{r1jk} R_{21jk} \qquad j,k = 0,1 \qquad (7.68)$$

$$-\varrho \int_{\bar{A}'} (\bar{p}^{+}_{1jk} - \bar{p}^{-}_{1jk})\, \bar{x}\, d\bar{x}\, d\bar{z} = \bar{l}'_{r1jk} R_{21jk} \qquad (7.69)$$

serve to determine the rudder force R_{21jk} and the coordinates $-\bar{l}'_{r1jk}$, $-\bar{t}'_{r1jk}$ of the point of the $\bar{x}'$, $\bar{z}'$-plane where it applies $j, k = 0, 1$. Then the quantity $T_{2jk} + R_{12jk}$ is determined by

$$-\varrho \int_{\bar{A}'} (\bar{p}^{+}_{1jk} + \bar{p}^{-}_{1jk})\, h_{\bar{x}'}\, d\bar{x}\, d\bar{z} = T_{2jk} + R_{12jk} \qquad j, k = 0, 1 \qquad (7.70)$$

and we have

$$S_{1jk} = 0 \qquad j, k = 0, 1.$$

The constant values of z_{1jk} and Θ_{21jk} must satisfy the equations

$$R_{32jk} - 2\varrho g \int_{\bar{L}'} (z_{1jk} + \bar{x}\,\Theta_{21jk})\, h\, d\bar{x} + \varrho \int_{\bar{A}'} (\bar{p}^{+}_{1jk} + \bar{p}^{-}_{1jk})\, h_{\bar{z}}\, d\bar{x}\, d\bar{z} = 0 \qquad (7.71)$$

$$\bar{t}'_t T_{2jk} + \bar{t}'_{r2jk} R_{12jk} - \bar{l}'_{r2jk} R_{32jk} + \varrho \int_{\bar{A}'} (\bar{p}^{+}_{1jk} + \bar{p}^{-}_{1jk}) [\bar{x}\, h_{\bar{z}'} - (\bar{z} - \bar{z}_c')\, h_{\bar{x}'}]\, d\bar{x}\, d\bar{z}$$

$$- 2\varrho g \left\{ \int_{\bar{L}'} (z_{1jk} + \bar{x}\,\Theta_{21jk})\, \bar{x}\, h\, d\bar{x} + \Theta_{21jk} \int_{\bar{A}'} (\bar{z} - \bar{z}_c')\, h\, d\bar{x}\, d\bar{z} \right\} = 0$$

$$j, k = 0, 1 \qquad (7.72)$$

Next consider the non-steady state case. Since we assumed constant propeller thrust and rudder force we have first for the determination of the quantities $\omega_{1jk}(t)$, $\Theta_{11jk}(t)$ and $\Theta_{31jk}(t)$, which appear in our integral equations, the differential equations

$$\rho \int_{\bar{A}'} (\bar{p}^{+}_{1jk} - \bar{p}^{-}_{1jk})\, d\bar{x}\, d\bar{z} = 0 \tag{7.73}$$

$$\rho \int_{\bar{A}'} (\bar{p}^{+}_{1jk} - \bar{p}^{-}_{1jk})\, \bar{z}\, d\bar{x}\, d\bar{z} = 0 \qquad j,k = 0,1 \tag{7.74}$$

$$\rho \int_{\bar{A}'} (\bar{p}^{+}_{1jk} - \bar{p}^{-}_{1jk})\, \bar{x}\, d\bar{x}\, d\bar{z} = 0 \tag{7.75}$$

together with the initial conditions

$$\omega_{1jk}(-\infty) = \dot{\Theta}_{11jk}(-\infty) = \Theta_{11jk}(-\infty) = \dot{\Theta}_{31jk}(-\infty) = \Theta_{31jk}(-\infty) = 0 \tag{7.76}$$

The other physical quantities $S_{1jk}(t)$ $z_{1jk}(t)$ and Θ_{21jk} are the solutions of the differential equations:

$$M_{100}\, \dot{s}_{1jk} = \rho \int_{\bar{A}'} (\bar{p}^{+}_{1jk} + \bar{p}^{-}_{1jk})\, h_{\bar{x}'}\, d\bar{x}\, d\bar{z} \tag{7.77}$$

and

$$M_{100}\, \ddot{z}_{1jk} = -\rho g \int_{\bar{L}'} (z_{1jk} + \Theta_{21jk}\bar{x})\, h\, d\bar{x} + \rho \int_{\bar{A}'} (\bar{p}^{+}_{1jk} + \bar{p}^{-}_{1jk})\, h_{\bar{z}'}\, d\bar{x}\, d\bar{z} \tag{7.78}$$

$$I_{2100}\, \ddot{\Theta}_{21jk} = -2\rho g \left\{ \int_{\bar{L}'} (z_{1jk} + \bar{x}\,\Theta_{21jk})\, \bar{x}\, h\, d\bar{x} + \Theta_{21jk} \int_{\bar{A}'} (\bar{z} - \bar{z}'_c)\, h\, d\bar{x}\, d\bar{z} \right\}$$

$$+ \rho \int_{\bar{A}'} (\bar{p}^{+}_{1jk} + \bar{p}^{-}_{1jk}) [\bar{x} h_{\bar{z}'} - (\bar{z} - \bar{z}'_c) h_{\bar{x}'}]\, d\bar{x}\, d\bar{z} \tag{7.79}$$

$$j,k = 0,1$$

which satisfy the initial conditions

$$s_{1jk}(-\infty) = \dot{z}_{1jk}(-\infty) = z_{1jk}(-\infty) = \dot{\Theta}_{21jk}(-\infty) = \Theta_{21jk}(-\infty) = 0 \quad j,k=0,1 \tag{7.80}$$

If it is desired to study motions with variable propeller thrust or rudder force we have only to take into consideration the propeller thrust or the rudder force as functions of t which satisfy the conditions

$$\lim_{t\to\pm\infty} T_{2jk}(t) = \lim_{t\to\pm\infty} R_{10jk}(t) = \lim_{t\to\infty} R_{21jk}(t) = \lim_{t\to\pm\infty} R_{32jk}(t) = 0 \quad j,k=0,1 \tag{7.81}$$

Now the coordinate $-\bar{\ell}'_{r1jk}$ and $-\bar{t}'_{r1jk}$ are also functions of t with limits

$$\lim_{t\to-\infty} \bar{\ell}'_{r1jk}(t) = \bar{\ell}'^{(-\infty)}_{r1jk}$$

$$\lim_{t\to+\infty} \bar{\ell}'_{r1jk}(t) = \bar{\ell}'^{(+\infty)}_{r1jk} \qquad j,k = 0,1$$

$$\lim_{t\to-\infty} \bar{t}'_{r1jk}(t) = \bar{t}'^{(-\infty)}_{r1jk}$$

$$\lim_{t\to+\infty} \bar{t}'_{r1jk}(t) = \bar{t}'^{(+\infty)}_{r1jk} \tag{7.82}$$

For the determination of $\omega_{1jk}(t)$, $\Theta_{11jk}(t)$ and $\Theta_{31jk}(t)$ we have the differential equations $(j,\ k = 0,\ 1)$:

$$g\int_{\bar{A}'}(\bar{p}^{+}_{1jk} - \bar{p}^{-}_{1jk})\, d\bar{x}\, d\bar{z} = R_{21jk}(t) \tag{7.83}$$

$$g\int_{\bar{A}'}(\bar{p}^{+}_{1jk} - \bar{p}^{-}_{1jk})(\bar{z} - \bar{z}'_c)\, d\bar{x}\, d\bar{z} = -\bar{t}'_{r1jk}(t)\, R_{21jk}(t) \quad j,k=0,1 \tag{7.84}$$

$$g\int_{\bar{A}'}(\bar{p}^{+}_{1jk} - \bar{p}^{-}_{1jk})\,\bar{x}\, d\bar{x}\, d\bar{z} = -\bar{\ell}'_{r1jk}(t)\, R_{21jk}(t) \tag{7.85}$$

together with the initial conditions (7.76). The other physical quantities $S_{1jk}(t)$, $z_{1jk}(t)$ and $\Theta_{21jk}(t)$ $j, k = 0, 1$ are solutions of the differential equations $j, k = 0, 1$:

$$M_{100}\, \dot{s}_{1jk} = g \int_{A'} (\bar{p}^{+}_{1jk} + \bar{p}^{-}_{1jk}) h_{\bar{x}'}\, d\bar{x}\, d\bar{z} + T_{2jk}(t) + R_{12jk}(t) \tag{7.86}$$

and

$$M_{100}\, \ddot{z}_{1jk} = -2 g g \int_{L'} (z_{1jk} + \bar{x}\,\Theta_{21jk}) h\, d\bar{x} + g \int_{A'} (\bar{p}^{+}_{1jk} + \bar{p}^{-}_{1jk}) h_{\bar{z}'}\, d\bar{x}\, d\bar{z} + R_{32jk}(t) \tag{7.87}$$

$$I_{2100}\, \ddot{\Theta}_{21jk} = -2 g g \left\{ \int_{L'} (z_{1jk} + \bar{x}\,\Theta_{21jk}) \bar{x}\, h\, d\bar{x} + \Theta_{21jk} \int_{A'} (\bar{z} - \bar{z}'_c) h\, d\bar{x}\, d\bar{z} \right\}$$
$$+ g \int_{A'} (\bar{p}^{+}_{1jk} + \bar{p}^{-}_{1jk}) [\bar{x} h_{\bar{z}'} - (\bar{z} - \bar{z}'_c) h_{\bar{x}'}]\, d\bar{x}\, d\bar{z}$$
$$+ \bar{t}'_t(t)\, T_{2jk}(t) + \bar{t}'_{t2jk}(t)\, R_{12jk}(t) - \bar{\ell}'_{t2jk}(t)\, R_{32jk}(t) \tag{7.88}$$

and the initial conditions (7.80).
In the general case

$$g(x,y) = g_0(y) + g_1(x,y) \qquad \text{with} \qquad \lim_{x \to \pm\infty} g_1(x,y) = 0$$
$$b_i(x,z) = b_{i0}(z) + b_{i1}(x,z) \qquad \text{with} \qquad \lim_{x \to \pm\infty} b_{i1}(x,z) = 0 \qquad i = 1,2$$

The solution can be found by superposition of the solution of the steady state problem corresponding to $g_0(y)$, $b_{10}(z)$, $b_{20}(z)$ and the solution of the non-steady state problem corresponding to $g_1(x,y)$, $b_{11}(x,z)$, $b_{21}(x,z)$.

The special problem, where there are a wavy bottom or wavy canal walls can be treated in the same manner, no additional difficulties arise. Thus the whole problem which appears in connection with the considered motion of a ship of Michell's type in a canal of variable breadth and depth is reduced to the solution of a system of integro-differential equations. Only if the problem is symmetrical with respect to the x,z-plane:

$$B_1 = B_2 = B \; ; \quad g(x,-y) = g(x,y) \quad ; \; b_1(x,z) = b_2(x,z) = b(x,z)$$

can the solution of our problem be given in an explicit form. This special case will be the subject of a future publication.

It is clear that our theory which is here given for the special case of a ship of Michell's type can without any difficulties be carried over to the cases of a planing ship or a ship of yacht-type hull. Only the boundary conditions on the ship's hull, the dynamic equations of motion of the ship and the integral equations for the surface distributions have to be reformulated. Especially in the case of the planing ship uniqueness theorems can be proven in a manner similar to that given by Peters and Stoker[9] for the motion of a planing ship in a seaway.

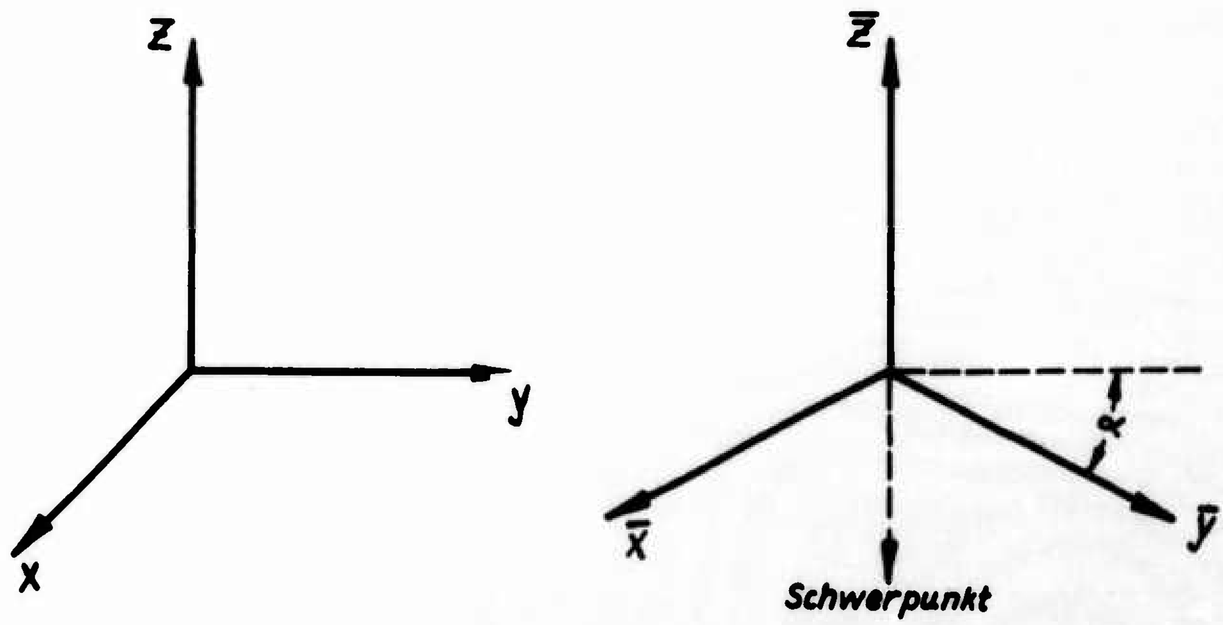

Fig. 1 : The fixed x,y,z- and the moving $\bar{x},\bar{y},\bar{z}$-coordinate system

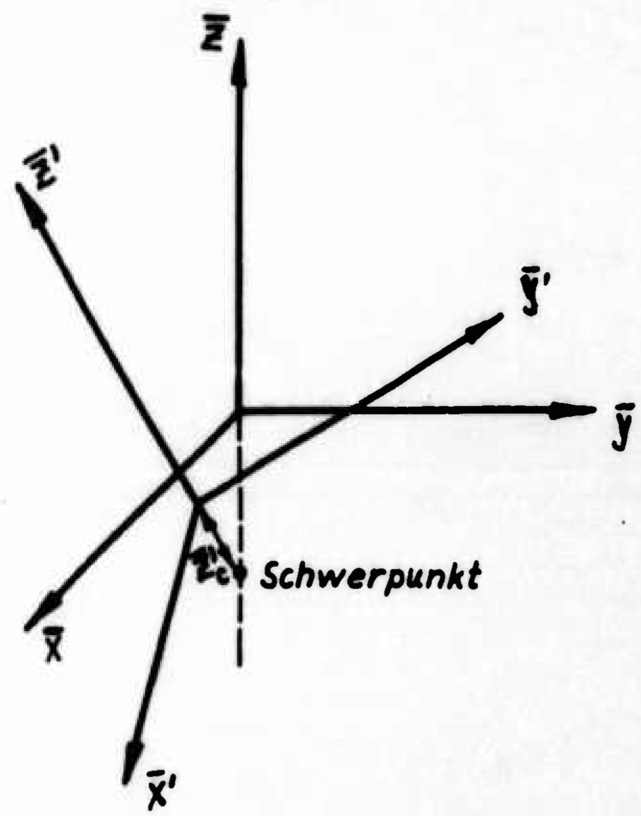

Fig. 2 : The moving $\bar{x},\bar{y},\bar{z}$- and $\bar{x}',\bar{y}',\bar{z}'$-coordinate systems

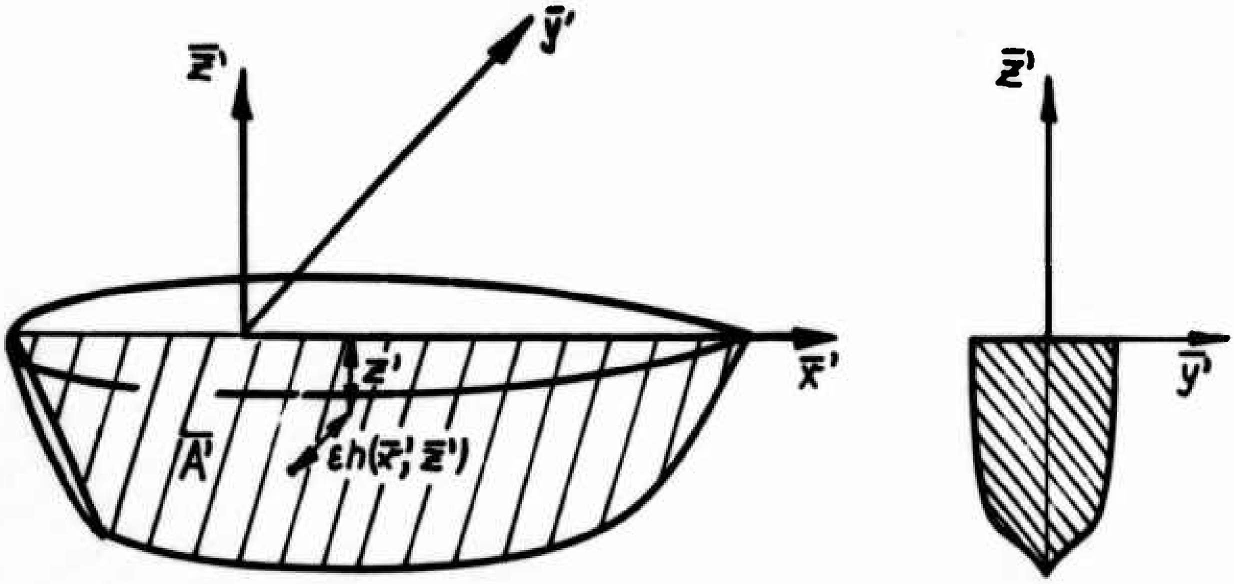

Fig. 3 : The equation of the ship's hull for a ship of Michell's type

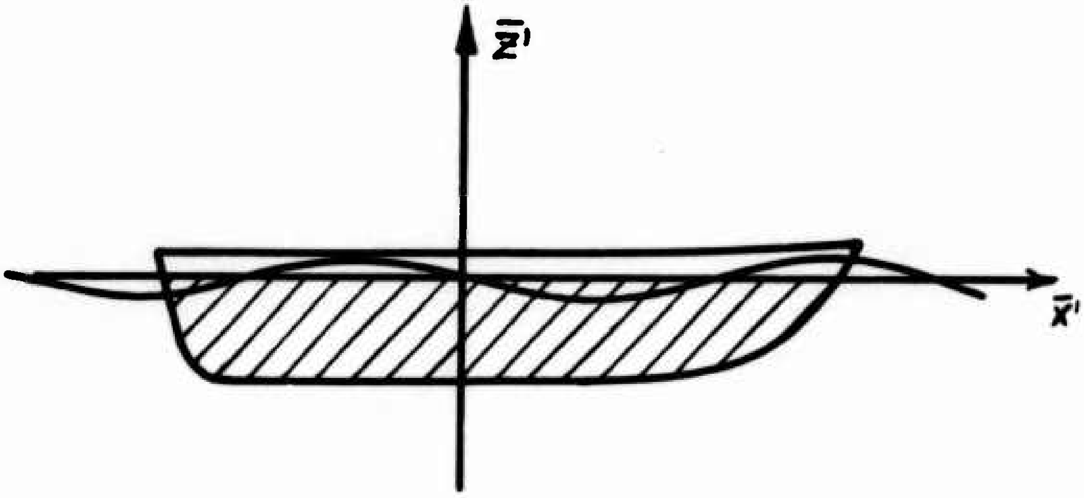

Fig. 4 : The region $\bar{A}'$

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ON VOSSERS' INTEGRAL

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Slender Ship Theories

Let us consider the uniform translation of a rigid body (ship) situated at our near the gravitational free surface of an infinite inviscid incompressible ocean which is at rest apart from the disturbance produced by the passage of the ship. This is a well formulated boundary value problem in hydrodynamics, but as it is non-linear, in order to approximate it by a mathematically tractable linearized problem it is generally necessary to assume that the disturbance produced by the ship at the free surface is small. In practice this means either that the ship is deeply submerged or the ship itself "small" in some sense.

For many years the only successful and mathematically sound approach to problems of the latter type has been that usually associated with the name of Michell(1898), and the theory of Michell, or "thin", ships has reached its highest stage of development in the work of Peters and Stoker (1957). Since thin ship theory is based upon the same type of approximation as is used in the thin aerofoil theory of aerodynamics, one is tempted to try another technique used by aerodynamicists for linearizing their problems, namely the theory of slender bodies. Recently several authors working independently (Vossers 1962, Maruo 1962, Tuck 1963) have been successful in establishing slender ship theories for the case of steady motion which are broadly on the same basic as aerodynamic slender body theory.

Vossers' approach is perhaps the least like the traditional aerodynamic method instituted by Ward (1949). Vossers formulates the linearized problem as an integral equation using Green's functions, and by constructing the approximate form of the Green's function in the slender region (i.e. in a region near to a straight line) occupied by the ship obtains a first approximation to the potential in this region. Although Vossers' analysis is unnecessarily complicated and contains numerous serious errors, his method has much to be said for it from the point of view of mathematical rigour, and it would seem worthwhile applying it to the original aerodynamic problem as well. In spite of errors Vossers obtains a wave resistance formula by integrating hull pressure which is only wrong by a factor of four.

Maruo represents the ship by a volume distribution of sources and then, by non-dimensionalising his problem with distances in cross-sectional planes stretched compared with axial distances, reduces the volume distribution to a line distribution (except at the ends of the ship). The strength of this line distribution may now be shown by arguments similar to those of Ward to be proportional to the axial derivative of immersed cross-sectional

area. The wave resistance follows from results of Havelock (1932), and Maruo obtains essentially the same resistance formula as Vossers, together with some new end-effect terms.

Perhaps a reason why the development of slender ship theories has lagged so long behind that of thin ship theories is that in the latter case the hull boundary condition may be approximately applied on the plane surface to which the ship shrinks as its beam tends to zero. whereas it is not mathematically feasible to apply boundary conditions on the line in three dimensions to which a slender body shrinks. This is why it is useful to consider the problem with a distorted co-ordinate system such as Maruo used, since in the new co-ordinate system the body does not shrink down to a line. However, in this "inner" region we can say little about how the flow behaves at infinity and to complete the solution it is necessary to also consider the original physical space with the ship shrunk down to a line. In this "outer" region the only possible disturbance is on the limiting line, so that the flow behaves like a line distribution of (Kelvin) sources of unknown density. The solution to the problem may now be completed by matching the behavior of the inner and outer solutions in a supposed common domain of validity.*

Thus in the inner region the disturbance potential becomes two dimensional in cross-sectional planes, and behaves at large distances r from the ship in such planes like a 2D line source of known strength; in fact

$$\phi \sim \frac{U}{\pi} S'(x) \log r + b(x) \tag{1}$$

where U is the speed of the ship, and $S(x)$ is the immersed cross-sectional area as a function of the axial co-ordinate x. The function "$b(x)$" is a constant in planes of constant x, and is arbitrary until matching takes place.

On the other hand (Tuck, 1963) the outer solution behaves for <u>small</u> r like

$$\begin{aligned}\phi \sim\ & a(x) \log r \\ & - \frac{1}{2} \int_{-\infty}^{\infty} da(\xi) \left\{ \operatorname{sgn}(x-\xi) \log 2\,|x-\xi| + \frac{\pi}{2} H_o\,(k(x-\xi)) \right. \\ & \left. + (2 + \operatorname{sgn}(x-\xi)) \frac{\pi}{2} Y_o(k|x-\xi'|) \right\}\end{aligned} \tag{2}$$

*This is a brief and inadequate symmary of a singular perturbation technique of wide applicability in applied mathematics. The method of inner and outer expansions has been mainly used till now for problems involving viscous flow at low Reynolds' number and a very careful rigorous statement of the princip involved is given by Lagerstrom and Cole (1955).

where sgn = ± 1 according to the sign of its argument and $\underset{\sim}{H}_o$, Y_o are Struve and Bessel Y functions respectively (Erdelyi, 1953, pp. 8, 37). That is, the outer solution behaves like a 2D line source of unknown strength $a(x)$ together with a function of x (and of the constant $k = g/U^2$) which is determined uniquely from $a(x)$. Now matching the inner and outer solutions together gives

$$a(x) = \frac{U}{\pi} S'(x)$$

$$b(x) = -\frac{U}{2\pi} \int_{-\infty}^{\infty} d\, S'(\xi) \left\{ \operatorname{sgn}(x-\xi) \log 2 |x-\xi| + \frac{\pi}{2} \underset{\sim}{H}_o (k(x-\xi)) \right.$$
$$\left. + (2 + \operatorname{sgn}(x-\xi)) \frac{\pi}{2} Y_o (k|x-\xi|) \right\} \qquad (4)$$

so that both solutions are now rendered complete and unique. Vossers has given a result similar to our Equation (1) but his formula for the function corresponding to $b(x)$ as given by (4) is incorrect.

The formula (4) has an interesting physical description. Let us consider separately the ranges of integration $\xi > x$ and $\xi < x$. In the former case the Struve and Y functions occur only in the particular combination "$\underset{\sim}{H}_o - Y_o$" which is well known (Erdelyi, p. 38) to be a monotone decreasing function of its (positive) argument. Hence this portion of the range of integration does not contribute to the wave-like character of the disturbance, which is what we should expect since it is a contribution from disturbance points behind the point x of observation (we have chosen axes such that the ship moves in the negative x direction). Further, the kernel of the integral in (4) is bounded as $\xi \to x +$, since the "log" term precisely cancels the logarithmic behavior of the Y_o function.

On the other hand, when $\xi < x$ the kernel contains a different combination of log, $\underset{\sim}{H}_o$, and Y_o , retaining all the oscillatory nature of the Bessel and Struve functions, and behaving logarithmically as $\xi \to x-$. Thus this portion of the range of integration, which corresponds to disturbance ahead of the observer, contributes most to both the wave-like and source-like nature of the flow near the ship. This description of the behavior of the function $b(x)$ has more than qualitative importance, since it can be shown (Tuck, 1963) that the amplitude of the transverse wave either in front of or behind the ship is proportional to $b'(x)$. In front of the ship we must have $\xi > x$ so that no waves are predicted, whereas far behind the ship we can predict from the asymptotic behavior of the H_o and Y_o functions that there is a wave system with wave number k and with amplitude decreasing like $x^{-\frac{1}{2}}$.

Vossers' Integral

The wave resistance may be found either from the inner potential by pressure integration along the hull, or from the outer potential by use of Havelock on source distributions, and the same answer is obtained in both cases. Thus by a pressure integration we have the wave resistance as

$$\begin{aligned} R &= -\rho \int_{-\infty}^{\infty} dS(x)\ (U\, b'(x)) \\ &= \rho\, U \int_{-\infty}^{\infty} dS'(x)\ b(x) \\ &= -\frac{1}{2}\rho\, U^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dS'(x)\ dS'(\xi)\ Y_0\ (k|x-\xi|) \end{aligned} \tag{5}$$

since only that part (involving Y_0) of the kernel in (4) which is an even function of its argument will contribute to the above integral. (The reason why Vossers is able to obtain the correct form of the resistance integral is that the error in his expression for $b(x)$ involves omission of some odd terms only from the kernel.) If it is assumed that $S(x)$ vanishes identically outside a finite interval, the length of the ship, then the above Stieltjes integral may be written as a finite Riemann integral involving the second derivative $S''(x)$; however, if $S'(x)$ is not continuous over the whole real axis, then extra terms will be introduced at the points of discontinuity, and Vossers supplies these explicitly for the case when $S'(x)$ does not vanish at the ends of the ship.

The formula (5) may be called the Vossers integral in dimensional form. Non-dimensional formulations are sometimes (but by no means always) more useful. There are innumerable methods of carrying out non-dimensionalisation, but in the present context the most meaningful is that based on displacement, as used by Maruo. For definiteness let us assume that the ship is of finite length L and that $S'(\pm \frac{L}{2}) = 0$, so that the end-effect terms do not occur. Then we put

$$\sigma(t) = \frac{L}{\Delta}\, S(\frac{Lt}{2}) \tag{6}$$

where Δ is the displacement,

$$\Delta = \int_{-L/2}^{L/2} S(x)\, dx \quad .$$

Hence we have

$$\int_{-1}^{1} dt\, \sigma\,(t) = 2 \tag{7}$$

In terms of $\sigma(t)$, Vossers' integral becomes

$$\frac{R}{\frac{1}{2}\rho\, U^2\, L^2} = \frac{8}{\pi}\left(\frac{\Delta}{L^3}\right)^2 f(F) \tag{8}$$

where we have defined a resistance coefficient $f(F)$ of the form

$$f(F) = -\frac{\pi}{2}\int_{-1}^{1}\int_{-1}^{1} dt_1\, dt_2\, \sigma''(t_1)\, \sigma''(t_2)\, Y_o\left(\frac{|t_1 - t_2|}{2F^2}\right) \tag{9}$$

with $F = \frac{U}{\sqrt{gL}} = \frac{1}{\sqrt{kL}}$ as the Froude number. There are other useful forms for the resistance; e.g. a "p's and q's" form is

$$f(F) = \int_{o}^{\infty} d\gamma\, p^2\left(\frac{\cosh\gamma}{2F^2}\right) + q^2\left(\frac{\cosh\gamma}{2F^2}\right) \tag{10}$$

where

$$p(k) + i\, q(k) = \int_{-1}^{1} d\, t\, e^{\,i\,k\,t}\, \sigma''(t) \tag{11}$$

is the Fourier Transform of $\sigma''(t)$. Also it is sometimes convenient to use a "hull function" form, with

$$f(F) = \frac{-\pi}{2}\int_{o}^{1} dv\, Y_o\left(\frac{v}{F^2}\right) H(v) \tag{12}$$

where

$$H(v) = 4\int_{-1+v}^{1-v} du\, \sigma''(u + v)\, \sigma''(u - v) \tag{13}$$

High Speed Approximations

The integrals (9) or (12) can easily be approximated for large values of F using known behavior of the Y_o function at small values of its argument (Erdelyi, p. 8). The result from (12) is

$$f(F) = f_o + f_1 F^{-4} + \frac{3}{16} F^{-8} \log F + f_2 F^{-8} + O(F^{-12} \log F) + \ldots \tag{14}$$

where

$$f_o = - \int_0^1 dv \log_e r \quad H(v) \tag{15}$$

$$f_1 = \frac{1}{4} \int_0^1 dv \log_e \; r \, v^2 H(v) \tag{16}$$

$$f_2 = - \frac{1}{64} \int_0^1 dv \log_e \; r \, v^4 H(v) + \frac{3}{32} \left(\frac{3}{2} + \log_e 2 - \gamma\right) \tag{17}$$

etc. It may easily be demonstrated (Tuck, 1963) that f_o and f_1 are non-negative, and that the contribution of the integral to f_2 and further terms is of ever diminishing importance compared with the constant terms involving Euler's constant γ, which occur in all f_n, $n \geq 2$.

Now the terms actually written down in the above series (14) already give information about the highest peak on the resistance curve, for (14) clearly breaks down at a value of $F(< 1)$ such that the term "$\frac{3}{16} F^{-8} \log F$" sends f negative, whereas as F tends to indinity, f tends to the non-negative value f_o from above (since f_1 is non-negative). In fact there are plausible grounds for estimating the position of the last peak as that value of F for which the "F^{-8}" terms cancel each other, i.e. at $F = F_o$, where

$$F_o \simeq e^{-\frac{16}{3} f_2} \tag{18}$$

Further, we may gain an even rougher, but universal, estimate of this peak by putting $f_2 \simeq \frac{3}{32} (\frac{3}{2} + \log 2 - \gamma) = 0.1515 ..$ in this formula, i.e. by neglecting the contribution of the integral to (17). This gives $F_o = 0.466..$; if exact values of f_2 are used in any particular case, somewhat higher values of F_o are obtained, in the range 0.45 to 0.5. This appears

to agree to the accuracy expected with experimental observations, and the fact that F_o does not appear in experiments to depend strongly on the shape of the hull is a reflection of the fact that f_2 is not much different from 0.1515 for practical ship shapes.

If further terms in the series (14) were used, the lower limit of F for which the formula is valid might well be pushed down to the practical operating range of ships; as it stands the formula (14) ceases to give sensible results for F below about 0.4. It is possible (Tuck, 1963) to make low speed approximations similar to those obtained by Kotik (Wehausen, 1957) for Michell's integral, but these seem to have no quantitative use; in any case the full Vossers integral itself is not expected to be valid for small F.

Minimisation at High Speed

Maruo (1963) has given a method for finding the shape $\sigma(t)$ which gives least drag in terms of Mathieu functions. It may be worth noticing that at high speed a very simple result (Tuck, 1963) is obtained for the minimisation of the coefficient f_o. The full kernel Y_o has now reduced to a log function, and the integral equation for the minimisation problem can be solved by use of Hilbert transforms, giving

$$\sigma(t) = \frac{8}{3\pi} (1-t^2)^{\frac{3}{2}} \tag{19}$$

(normalised by (7)), with a corresponding resistance at infinite Froude number of $f_o = 32$. This is the least possible value of f_o; the same ship would give a resistance of

$$f(F) = 64 \int_0^\infty d\gamma \left[J_2\left(\frac{\cosh\gamma}{2F^2}\right)\right]^2 \tag{20}$$

at finite F, but the latter is of course not by any means the least possible value.

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DISCUSSION

by G. R. G. Lewison

The experimental confirmation of Mr. Tuck's theory as to wave resistance is a little distant, as indicated by results described yesterday, but I should like to mention some other results which I have computed and may be of interest.

Firstly, the function b, of Equation (4), or rather its derivative (b') with respect to x, has been calculated for a range of speeds. The general behavior is much as described by Mr. Tuck, and b' is very small and non-oscillatory ahead, and oscillates with decreasing amplitude astern There are discontinuities of slope at the ends, $x = \pm 1$; within the body [the function resembles the wave profile as experimentally observed,* when corrected by two small terms dependant upon the section area derivative]. This resemblance is more noticable as speed increases, and is fair at $f \sim .3$.

Secondly, the position of the last hump, for my choice of model, is given quite well by the calculations of page 10; in fact $f \simeq .49$ theoretically, in close agreement with the experimental value for one model of $F = .47$.

Thirdly, the calculated wave resistance coefficient, proportional to Tuck's f(F) of Equation (14), seems to behave very much as indicated by this equation, at high speeds, and to approach a minimum value, which remains quite large, as the speed increases further.

* As mentioned yesterday by Mr. Tuck in his discussion of Professor Takahei's paper.

by J. N. Newman

I should like to outline an alternative approach to the wave resistance of a slender body. This is based upon the relatively simple analysis of the far-field potential through Green's theorem. For a point in the interior of the fluid we have

$$\phi = \frac{1}{4\pi} \iint (G\phi_n - \phi G_n)\, dS + O(\varepsilon^3 \log \varepsilon)$$

where the integral is over the submerged surface of the body, and the Green's function G represents the potential of a source satisfying the linearized free surface condition. Now we can apply the above relation to a point in the far-field, whence G and G_n are non-singular and $O(1)$. On the other hand, from the solution of the first order problem in the near field, $\phi = O(\varepsilon^2 \log \varepsilon)$ while $\phi_n = O(\varepsilon)$. Thus for points in the far-field the above equation reduces to the simple relation

$$\phi = \frac{1}{4\pi} \iint G\phi_n\, dS + O(\varepsilon^3 \log \varepsilon)$$

and the first-order far-field potential consists simply of a source distribution of known strength. Since the far-field potential is sufficient for the determination of the wave resistance, the above relation may be used for this purpose. Moreover it suggests an explanation of the fact that the first order slender body wave resistance is so closely related to Michell's integral, for the Mitchell potential for a thin ship is also a source distribution of strength ϕ_n.

The above approach can also be carried out to second order, providing a relation for the far-field second-order potential in terms of the first order potential in the near field.

by H. Maruo

I appreciate Mr. Tuck and Mr. Joosen who obtained independently and simultaneously the right expression for the velocity potential of a slender ship. When Dr. Vossers and I got the wave resistance formula for the slender ship independently last year, there was a difference between our results unfortunately. However, the results presented here show a thorough agreement, in spite of the methods by which the formula is derived are different. Therefore we can conclude that the slender ship theory can claim its right as a consistent theory. I wish to point out some of the difficulties when the slender ship theory is considered. That is the higher singularity which appear in the kernel of the integral. In the case of the ship form which has finite angle of entrance, a special treatment is needed for the end singularities in order to secure a converging integral. For a ship form with pointed nose, this difficulty does not appear in the calculation of the wave resistance, but if we need to discuss the surface profile, singularities occur at both ends. Though the slender ship theory looks nicely as a linearized theory, it does not necessarily so if a practical application is considered, as we can see in Mr. Lewison's result. One of the reasons is the fact that the theory assumes free surface as a rigid plane. The basic idea of the slender ship is to regard the fluid flow near the body surface as two-dimensional, and this assumption is so strong as to be satisfied by the second approximation as well. The assumption of the rigid free surface however is much weaker than it. It does not hold in the second approximation. When the second approximation is considered in order to obtain a better agreement with the actual phenomenon, a difficulty appears again from the higher singularity. Converging form can be obtained only for a very sharp pointed form with cusps. This case is by no means practical. This fact suggests that the perturbation scheme is not regular. Some special techniques are needed to handle the end singularities. Multipole expansion cannot be applied to the slender ship with finite forward velocity. Anyway, the theory presented here is the only self-consistant linearized theory for a slender ship, and it should be highly evaluated.

AUTHOR'S REPLY

Professor Maruo's remarks are very pertinent. Time did not allow discussion of them in my presentation, but in my thesis (Tuck, 1963) I investigated in detail all the questions raised by Professor Maruo. In particular I obtained a second approximation with the properties described by him, viz. it is still derived from a two-dimensional potential function but not from a rigid wall boundary condition and it is indeed very badly behaved at the ends except for cusped ships. These end-effects, which are analogous to those experienced in thin aerofoil theory, clearly need special treatment in some sort of singular perturbation scheme.

Dr. Newman's method of constructing a slender ship theory is very useful, especially insofar as it indicates the essential similarities (and differences) between thin and slender ship theories. The application of his method to finding a second order potential would, however, be very difficult since the second order potential depends essentially on non-linear effects from the free surface condition near the ship. Thus a naive approach which assumes the usual linearized outer free surface condition to be valid everywhere does not work (in spite of the fact that the linearized condition is valid even for the second approximation, except in the inner region near the ship).

The fact that the qualitative and asymptotic descriptions given in my paper seem to agree well with calculations for a specific hull form, as reported by Mr. Lewison, suggests that simple approximations to the full slender ship results (such as the high speed asymptotics of Equation (14)) may be as useful as detailed computations. One should not in any case expect much quantitative agreement with experiment from a theory which itself requires severe approximations in its derivation, and the chief merit of slender ship theory may well be the simplicity of some of the final results which may be utilized in qualtative deductions.

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THE VELOCITY POTENTIAL AND WAVE RESISTANCE ARISING FROM THE MOTION OF A SLENDER SHIP

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SUMMARY

The paper consists of two parts. The first part deals with the steady forward motion of a slender ship. The velocity potential and wave resistance arising from the motion are calculated and the problem of minimum resistance is discussed and some numerical examples are given. In the second part a slender ship oscillating at zero forward speed is considered. Again the velocity potential is calculated. An integral equation for the corresponding two dimensional problem is among the results of this part.

The basic ideas are essentially the same as those in Vossers's work, but it appears that Vossers's method of calculation can be improved and that his results are not complete.

INTRODUCTION

Attention has been paid recently to the wave resistance of ships, having small beam and draft compared with the length and advancing at a Froude number of order one. Vossers(1) started from the complete linearized three dimensional solution of the velocity potential by means of Greens theorem and expanded this expression in terms of the small parameter σ , representing the ratio of both draft and beam to the length of the ship.

The starting point of Maruo(2) was Havelock's formula for the wave resistance of a source distribution representing the ship's hull. This formula was developed with respect to σ as well. The total source strength of every section and the source density at stern and bow appear in the final result. These unknown quantities are deduced from the boundary condition on the hull. The two terms depending on the source density at stern and bow are missing in Vosser's formula, which suggests that his expansion with respect to σ might be imcomplete. The purpose of the first part of this paper is to derive a formula for the velocity potential and the wave resistance in a more accurate way. The starting point is the formulation in wave source distribution using the Green function for the free surface condition.

The problem of a slender ship oscillating at zero forward speed is attacked in the second part of the paper with the same method as used

in the first part. Different results are obtained for small and for large frequencies. For the special case of a slender body of revolution Ursell[3] solved this problem recently by means of axial line distributions and Fourier transforms. The first terms in his expansion are also found with the method in this paper.

It is shown that for large frequencies the potential is in a first approximation the same as the two dimensional potential. For this case an integral equation is derived for the unknown distribution function.

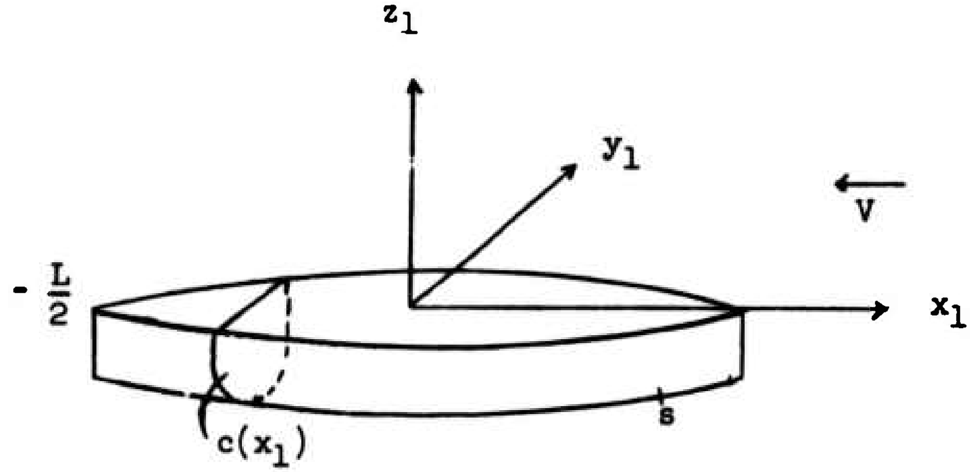
It is believed that the formulation as will be discussed in the following sections can be useful for the solution of the full problem of the slender ship oscillating at forward speed.

PART I

THE VELOCITY POTENTIAL AND WAVE RESISTANCE FOR STEADY FORWARD MOTION

1. The Velocity Potential

In the co-ordinate system used in the following the x_1, y_1 plane coincides with the free surface. The origin moves with the ship speed V in the same direction as the ship and the z-axis is taken positive in upward direction.



The hull surface is assumed to be of the form:

$$y_1 = f_1(x_1, z_1)\,\mathrm{sgn}\,y_1 \tag{1.1}$$

with

$$f_1(\tfrac{L}{2}, 0) = f_1(-\tfrac{L}{2}, 0) = 0$$

The length of the ship is L, the Beam B and the draft T. The draft at stern and bow is d_1. The form of the section is denoted by $C(x_1)$.

The velocity potential $\phi\,(x_1, y_1, z_1)$ should satisfy the equations:

$$\begin{aligned} &\phi_{x_1x_1} + \phi_{y_1y_1} + \phi_{z_1z_1} = 0 \quad , \quad z_1 \leq 0 \\ &V^2\phi_{x_1x_1} + g\phi_{z_1} = 0 \quad , \quad z_1 = 0 \end{aligned} \tag{1.2}$$

where differentiations are denoted by subscripts.

The boundary condition on the hull is

$$\frac{\partial \phi}{\partial n} = V \frac{\partial f_1}{\partial x_1}\,\mathrm{sgn}\,y_1$$

where n is a co-ordinate in the direction normal to the surface S.

The following dimensionless quantities are introduced

$$\begin{aligned} &x_1 = \frac{L}{2}\xi_1 \qquad y_1 = \sigma\frac{L}{2}\eta_1 \qquad z_1 = \sigma\frac{L}{2}\zeta_1 \qquad v_1 = \sigma\frac{L}{2}n \\ &f_1(x_1, z_1)\,\mathrm{sgn}\,y_1 = \sigma\frac{L}{2}f(\xi_1, \zeta_1) \quad \beta_o = \frac{2V^2}{gL} \qquad \sigma = \frac{B}{L} \\ &\phi(x_1, y_1, z_1) = Vx_1 + \sigma^2 V\frac{L}{2}\varphi(\xi_1, \eta_1, \zeta_1) \end{aligned} \tag{1.3}$$

$\sigma^2 V\frac{L}{2}\varphi(\xi_1, \eta_1, \zeta_1)$ is the potential due to the disturbance of the uniform motion of the fluid. The most important term occurring in the slender body theory is the zero order term in σ of $\varphi(\xi_1, \eta_1, \zeta_1)$. The derivation of this term is the purpose of the following procedure.

The conditions for $\varphi(\xi_1, \eta_1, \zeta_1)$ are

$$\sigma^2\varphi_{\xi_1\xi_1} + \varphi_{\eta_1\eta_1} + \varphi_{\zeta_1\zeta_1} = 0 \quad , \quad \zeta_1 \leq 0 \tag{1.4}$$

$$\sigma\,\beta_o\,\varphi_{\xi_1\xi_1} + \varphi_{\zeta_1} = 0 \quad , \quad \zeta_1 = 0 \tag{1.5}$$

$$\frac{\partial\varphi}{\partial\nu_1} = \varphi_{\eta_1} + \varphi_{\zeta_1} f_{\zeta_1} + O(\sigma) = f_{\xi_1} \quad , \quad \eta_1 = f(\xi_1, \zeta_1) \tag{1.6}$$

The potential φ can be written in the form

$$\varphi(\xi_1, \eta_1, \zeta_1) = \int_{-1}^{1} d\xi \int_{C(\xi)} F(\xi,\zeta) d\zeta \; G(\xi,\eta,\zeta;\xi_1,\eta_1,\zeta_1) \tag{1.7}$$

where $G(\xi,\eta,\zeta;\xi_1,\eta_1,\zeta_1)$ is Green's function for (1.4) satisfying the free surface condition (1.5). The unknown distribution $F(\xi,\zeta)$ is to be determined from the boundary condition (1.6).

Following a suggestion of Prof. Timman of the Technical University at Delft, such a form for G was chosen that a splitting up of the integration over ξ and ζ in more than two areas as was done in the work of Vossers,(1) can be avoided.

With this in mind and neglecting terms of order higher than zero (cf. Appendix A-3) is taken

$$\begin{aligned} G(\xi,\eta,\zeta;\xi_1,\eta_1,\zeta_1) = & \frac{1}{\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-f)^2+\sigma^2(\zeta_1-\zeta)^2}} \\ & + \frac{1}{\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-f)^2+\sigma^2(\zeta_1+\zeta)^2}} \\ & + \frac{2}{\pi} \mathcal{R} \, i \frac{\partial}{\partial\xi} \int_{-\infty}^{\infty} d\tau \int_M \frac{\beta_o}{\beta_o p-1} e^{\frac{1}{2}p\sigma(\zeta_1+\zeta) - p\sigma a\cosh 2\tau + pb(\xi_1-\zeta)\sinh\tau + ipc(\xi_1-\xi)\cosh\tau} dp \end{aligned} \tag{1.8}$$

with

$$\begin{aligned} a &= \frac{1}{2}\sqrt{(\zeta_1+\zeta)^2 + (\eta_1-f)^2} \\ b &= -\left\{ \frac{2a-(\zeta_1+\zeta)}{4a} \right\}^{1/2} \operatorname{sgn}(\eta_1-f) \\ c &= \left\{ \frac{2a+(\zeta_1+\zeta)}{4a} \right\}^{1/2} \end{aligned} \tag{1.9}$$

$\mathcal{R}$ means the real part and the contour M passes the pole in the upper half plane.

The potential (1.7) becomes:

$$\varphi(\xi_1,\eta_1,\zeta_1) = \int_{-1}^{1} d\xi \int_{C(\xi)+\overline{C}(\xi)} F(\xi,\zeta) \frac{d\zeta}{\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-f)^2+\sigma^2(\zeta_1-\zeta)^2}}$$

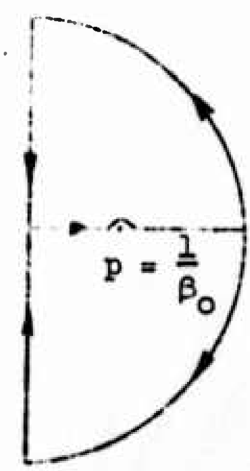
$$+ \frac{2}{\pi} i \int_{-1}^{1} d\xi \int_{C(\xi)} F(\xi,\zeta)\, d\zeta \frac{\partial}{\partial\xi} \int_{-\infty}^{\infty} d\tau \int_{M} \frac{\beta_o}{\beta_o p-1} \exp\,[\tfrac{1}{2}p\sigma(\zeta_1+\zeta)-p\sigma a\cosh 2\tau+pb(\xi_1-\xi)\sinh\tau + ipc(\xi_1-\xi)\cosh\tau]\, dp \tag{1.10}$$

where $\overline{C}(\xi)$ is the contour obtained by reflecting $C(\xi)$ with respect to the ξ-O-η plane.

The first integral of (1.10) can be considered as the potential associated with the motion of a body in an unbounded medium. The slender body theory for that case is well known and can be found f.i. in Ward.[4] Because this part of the velocity potential gives no contribution to the wave resistance, the discussion is restricted to the second integral:

$$\varphi_1(\xi_1,\eta_1,\zeta_1) = -\frac{2}{\pi}\kappa_o\, i \int_{-1}^{1} d\xi \int_{C(\xi)} F(\xi,\zeta) d\zeta \frac{\partial}{\partial\xi} \int_{-\infty}^{\infty} d\tau \int_{M} \frac{\beta_o}{\beta_o p-1} \exp[\tfrac{1}{2}p\sigma(\zeta_1+\zeta)-p\sigma a\cosh 2\tau+pb(\xi_1-\xi)\sinh\tau+ipc(\xi_1-\xi)\cosh\tau]\, dp \tag{1.11}$$

The integration contour M can be changed by closing the contour at the upper side if $(\xi_1-\xi) > 0$ and at the lower side if $(\xi_1-\xi) < 0$.



The separation of the ξ-interval is made in order to ensure the convergence of the integrals on the lines with completely imaginary argument.

$$\varphi_1(\xi_1,\eta_1,\zeta_1) = \frac{2}{\pi}\,\mathrm{Re}\int_{-1}^{\xi_1} d\xi \int_{C(\xi)} F(\xi,\zeta)d\zeta\,\frac{\partial}{\partial\xi}\int_{-\infty}^{\infty} d\tau \int_{0}^{\infty} \frac{\beta_0}{i\beta_0 p-1}$$
$$\exp\left[\tfrac{i}{2}p\sigma(\zeta_1+\zeta)-ip\sigma a\cosh 2\tau+ipb(\xi_1-\xi)\sinh\tau-pc(\xi_1-\xi)\cosh\tau\right] dp$$
$$+\frac{2}{\pi}\,\mathrm{Re}\int_{\xi_1}^{1} d\xi \int_{C(\xi)} F(\xi,\zeta)d\zeta\,\frac{\partial}{\partial\xi}\int_{-\infty}^{\infty} d\tau \int_{0}^{\infty} \frac{\beta_0}{i\beta_0 p+1}$$
$$\exp\left[-\tfrac{i}{2}p\sigma(\zeta_1+\zeta)+ip\sigma a\cosh 2\tau-ipb(\xi_1-\xi)\sinh\tau+pc(\xi_1-\xi)\cosh\tau\right] dp$$
$$-4\,\mathrm{Re}\int_{\xi_1}^{1} d\xi \int_{C(\xi)} F(\xi,\zeta)d\zeta\,\frac{\partial}{\partial\xi}\int_{-\infty}^{\infty}$$
$$\exp\left[\frac{\sigma}{2\beta_0}(\zeta_1+\zeta)-\frac{\sigma}{\beta_0}a\cosh 2\tau+\frac{b}{\beta_0}(\xi_1-\xi)\sinh\tau+\frac{ic}{\beta_0}(\xi_1-\xi)\cosh\tau\right] d\tau \qquad (1.12)$$

By partial integration with respect to ξ and deleating terms of higher order than the zero-th, is obtained:

$$\varphi_1(\xi_1,\eta_1,\zeta_1) = \frac{2}{\pi}\,\mathrm{Re}\int_{C(-1)} F(-1,\zeta)d\zeta\int_{-\infty}^{\infty} d\tau \int_{0}^{\infty} \frac{\beta_0 \exp}{1-i\beta_0 p}\left[\tfrac{i}{2}p\sigma(\zeta_1+\zeta)-ip\sigma a\cosh 2\tau\right.$$
$$+ipb(\xi_1+1)\sinh\tau$$
$$\left.-pc(\xi_1+1)\cosh\tau\right] dp$$
$$+\frac{2}{\pi}\,\mathrm{Re}\int_{C(1)} F(1,\zeta)d\zeta\int_{-\infty}^{\infty} d\tau \int_{0}^{\infty} \frac{\beta_0 \exp}{1+i\beta_0 p}\left[-\tfrac{i}{2}p\sigma(\zeta_1+\zeta)+ip\sigma a\cosh 2\tau\right.$$
$$-ipb(\xi_1-1)\sinh\tau$$
$$\left.+pc(\xi_1-1)\cosh\tau\right] dp$$
$$-\frac{2}{\pi}\,\mathrm{Re}\int_{C(\xi_1)} F(\xi_1,\zeta)d\zeta\int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} \frac{i\,\exp}{p+\frac{i}{\beta_0}}\left[\tfrac{i}{2}p\sigma(\zeta_1+\zeta)-ip\sigma a\cosh 2\tau\right] dp$$

$$+ \frac{2}{\pi} \mathcal{R}e \int_{-1}^{\xi_1} d\xi \int_{C(\xi)} F_\xi(\xi,\zeta) d\zeta \int_{-\infty}^{\infty} d\tau \int_0^{\infty} \frac{\beta_0 \exp}{1-i\beta_0 p} [ipb(\xi_1-\xi)\sinh\tau - pc(\xi_1-\xi)\cosh\tau] \, dp$$

$$\cdot \frac{2}{\pi} \mathcal{R}e \int_{\xi_1}^{1} d\xi \int_{C(\xi)} F_\xi(\xi,\zeta) d\zeta \int_{-\infty}^{\infty} d\tau \int_0^{\infty} \frac{\beta_0 \exp}{1+i\beta_0 p} \; ipb(\xi_1-\xi)\sinh\tau + pc(\xi_1-\xi)\cosh\tau] \, dp$$

$$- 4 \int_{C(\xi_1)} F(\xi_1-\zeta) \, d\zeta \int_{-\infty}^{\infty} \exp [\frac{\sigma}{2\beta_0}(\zeta_1+\zeta) - \frac{\sigma}{\beta_0} a\cosh 2\tau] \, d\tau$$

$$+ 4 \int_{C(1)} F(1,\zeta) \, d\zeta \int_{-\infty}^{\infty} \exp [\frac{\sigma}{2\beta_0}(\zeta_1+\zeta) - \frac{\sigma}{\beta_0} a\cosh 2\tau + \frac{b}{\beta_0}(\xi_1-1)\sinh\tau$$

$$+ ic(\xi_1-1)\cosh\tau] \, d\tau$$

$$- 4 \int_{\xi_1}^{1} d\xi \int_{C(\xi)} F_\xi(\xi,\zeta) d\zeta \int_{-\infty}^{\infty} \exp [\frac{b}{\beta_0}(\xi_1-\xi)\sinh\tau + i\frac{c}{\beta_0}\cosh\tau] \, d\tau \tag{1.13}$$

After transformation of the last two integrals in (1.13) according to Appendix A3, using the representation

$$\int_{-\infty}^{\infty} \exp [ipb(\xi_1-\xi)\sinh\tau - pc(\xi_1-\xi)\cosh\tau] \, d\tau = 2K_0(p|\xi_1-\xi|) \tag{1.14}$$

and neglecting terms of higher order is obtained:

$$\varphi_1(\xi_1,\eta_1,\zeta_1) = \frac{2}{\pi} \mathcal{R}e \int_{C(-1)} F(-1,\zeta) d\zeta \int_{-\infty}^{\infty} d\tau \int_0^{\infty} \frac{\beta_0}{1-i\beta_0 p} \times$$

$$\exp [\tfrac{1}{2} p\sigma(\zeta_1+\zeta) - ip\sigma a\cosh 2\tau + ipb(\xi_1+1)\sinh\tau - pc(\xi_1+1)\cosh\tau] \, dp$$

$$+ \frac{2}{\pi} Re \int_{C(1)} F(1,\zeta)d\zeta \int_{-\infty}^{\infty} d\tau \int_{0}^{\infty} \frac{\beta_o}{1+i\beta_o p} \quad x$$

$$\exp[-\tfrac{i}{2}p\sigma(\zeta_1+\zeta)+ip\sigma a\cosh 2\tau - ipb(\xi_1-1)\sinh\tau + pc(\xi_1-1)\cosh\tau]\ dp$$

$$+ 4\, Re \int_{C(1)} F(1,\zeta)d\zeta \int_{-\infty}^{\infty} e^{\frac{\sigma}{\beta_o}(\zeta_1+\zeta)\cosh^2\tau + i\,\frac{\sigma}{\beta_o}(\eta_1-f)\sinh\tau\cosh\tau + \frac{i}{\beta_o}(\xi_1-1)\cosh\tau} d\tau$$

$$+ \frac{4\beta_o}{\pi} \int_{-1}^{\xi_1} d\xi \int_{C(\xi)} F_\xi(\xi,\zeta)d\zeta \int_{0}^{\infty} \frac{K_o(p|\xi_1-\xi|)}{1+\beta_o^2p^2}\ dp$$

$$- \frac{4\beta_o}{\pi} \int_{\xi_1}^{1} d\xi \int_{C(\xi)} F_\xi(\xi,\zeta)d\zeta \int_{0}^{\infty} \frac{K_o(p|\xi_1-\xi|)}{1+\beta_o^2p^2}\ dp$$

$$- 4\, Re \int_{\xi_1}^{1} d\xi \int_{C(\xi)} F_\xi(\xi,\zeta)d\zeta \int_{-\infty}^{\infty} e^{\frac{i}{\beta_o}(\xi_1-\xi)\cosh\tau}\ d\tau \qquad (1.15)$$

Denoting $\int_{C(\xi)} F(\xi,\zeta)d\zeta$ by $S(\xi)$ and with the aid of the relation (1.14) together with

$$Re \int_{-\infty}^{\infty} e^{\frac{i}{\beta_o}(\xi_1-\xi)\cosh\tau}\ d\tau = -\pi\, Y_o\left(\frac{|\xi-\xi|}{\beta_o}\right) \qquad (1.16)$$

and

$$\int_{0}^{\infty} \frac{\beta_o K_o(p|\xi_1-\xi|)}{1+\beta_o^2p^2}\ dp = \frac{\pi^2}{4}\left\{ H_o\left(\frac{|\xi_1-\xi|}{\beta_o}\right) - Y_o\left(\frac{|\xi_1-\xi|}{\beta_o}\right)\right\}$$

where H_o is the Struve function and Y_o the second Bessel function, for the potential in a point ξ_1, at some distance from stern and bow is obtained:

$$\varphi_1(\xi_1,\eta_1,\zeta_1) = \varphi_1(\xi_1) = -4\pi\, S(1)\, Y_o(\frac{1-\xi_1}{\beta_o}) + 4\pi \int_{\xi_1}^{1} S_\xi(\xi)\, Y_o(\frac{|\xi_1-\xi|}{\beta_o})\, d\xi$$

$$+ \pi S(1)\left\{H_o(\frac{1-\xi_1}{\beta_o}) - Y_o(\frac{1-\xi_1}{\beta_o})\right\} + \pi S(-1)\left\{H_o(\frac{1+\xi_1}{\beta_o}) - Y_o(\frac{1+\xi_1}{\beta_o})\right\}$$

$$+ \pi\int_{-1}^{\xi_1} S_\xi(\xi)\left\{H_o(\frac{|\xi_1-\xi|}{\beta_o}) - Y_o(\frac{|\xi_1-\xi|}{\beta_o})\right\}d\xi$$

$$- \pi\int_{\xi_1}^{1} S_\xi(\xi)\left\{H_o(\frac{|\xi_1-\xi|}{\beta_o}) - Y_o(\frac{|\xi_1-\xi|}{\beta_o})\right\}d\xi \tag{1.17}$$

The velocity potential at bow and stern can be written as

$$\varphi_1(1,\eta,\zeta_1) = -4\int_{C(1)} F(1,\zeta)\ln\left\{\frac{\gamma\sigma}{4\beta_o}\sqrt{(\zeta_1+\zeta)^2 + (\eta_1-f)^2}\right\} d\zeta$$

$$+ \pi S(-1)\left\{H_o(\frac{2}{\beta_o}) - Y_o(\frac{2}{\beta_o})\right\} + \pi\int_{-1}^{1} S_\xi(\xi)\left\{H_o(\frac{1-\xi}{\beta_o}) - Y_o(\frac{1-\xi}{\beta_o})\right\} d\xi \tag{1.18}$$

and

$$\varphi_1(-1,\eta_1,\zeta_1) = 4\int_{C(-1)} F(-1,\zeta)\ln\left\{\frac{\gamma\sigma}{4\beta_o}\sqrt{(\zeta_1+\zeta)^2+(\eta_1-f)^2}\right\} d\zeta - 4\pi S(1)\, Y_o(\frac{2}{\beta_o})$$

$$+ 4\pi\int_{-1}^{1} S_\xi(\xi)\, Y_o(\frac{1+\xi}{\beta_o})d\xi + \pi S(1)\left\{H_o(\frac{2}{\beta_o}) - Y_o(\frac{2}{\beta_o})\right\}$$

$$- \pi\int_{-1}^{1} S_\xi(\xi)\left\{H_o(\frac{1+\xi}{\beta_o}) - Y_o(\frac{1+\xi}{\beta_o})\right\} d\xi \tag{1.19}$$

where the relations are used

$$\lim_{\xi_1 \to 1} \mathrm{Re} \int_0^\infty \frac{\beta_0 e^{-\frac{i}{2} p\sigma(\zeta_1+\zeta)+p\sigma a\cosh 2\tau - ipb(\xi_1-1)\sinh\tau + pc(\xi_1-1)\cosh\tau}}{1+i\beta_0 p} dp = 0$$

$$\lim_{\xi_1 \to -1} \mathrm{Re} \int_0^\infty \frac{\beta_0 e^{\frac{i}{2} p\sigma(\zeta_1+\zeta)-ip\sigma a\cosh 2\tau + ipb(\xi_1+1)\sinh\tau - pc(\xi_1+1)\cosh\tau}}{1-i\beta_0 p} dp = -2\pi e^{-\frac{\sigma}{\beta_0} a\cosh} + O(\sigma) \tag{1.20}$$

and $K_0\left(\frac{\sigma}{2\beta_0}\sqrt{(\zeta_1+\zeta)^2+(\eta_1-f)^2}\right) = -\ln\left\{\frac{\gamma\sigma}{4\beta_0}\sqrt{(\zeta_1+\zeta)^2+(\eta_1-f)^2}\right\} + O(\sigma\ln\sigma)$

The expressions for the potential at stern and bow differ from the formulas found by Vossers and only the first two terms of formula (1.17) appear in his results.

Hence the total velocity potential becomes:

$$\varphi(\xi_1,\eta_1,\zeta_1) = \int_{-1}^{1} d\xi \int_{C(\xi)+\bar{C}(\xi)} F(\xi,\zeta) \frac{d\zeta}{\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-f)^2+\sigma^2(\zeta_i-\zeta)^2}} - \varphi_1(\xi_1,\eta_1,\zeta_1) \tag{1.21}$$

From (1.17), (1.18) and (1.19) it follows that

$$\frac{\partial\varphi_1}{\partial\nu} = 0$$

Therefore the boundary condition (1.16) becomes:

$$\frac{\partial}{\partial\nu_1}\int_{-1}^{1} d\xi \int_{C(\xi)+\bar{C}(\xi)} F(\xi,\zeta) \frac{d\zeta}{\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-f)^2+\sigma^2(\zeta_1-\zeta)^2}} = f_{\xi_1}(\xi_1,\zeta_1) \tag{1.22}$$

Taking into account only the zero order term, yields

$$\frac{\partial}{\partial\nu_1}\int_{C(\xi_1)+\bar{C}(\xi_1)} F(\xi_1,\zeta)\ \ln\sqrt{(\eta_1-f)^2+(\zeta_1-\zeta)^2}\ d\zeta = -\frac{1}{2} f_{\xi_1}(\xi_1,\zeta_1) \tag{1.23}$$

Integrating both parts of the equation with respect to ζ_1 over the contour and changing the order of integration the following is obtained

$$\int\limits_{C(\xi_1)+\bar{C}(\xi_1)} F(\xi_1,\zeta)d\zeta \int\limits_{C(\xi_1)+\bar{C}(\xi_1)} \frac{\partial}{\partial \nu_1} \ln\sqrt{(\eta_1-f)^2+(\zeta_1-\zeta)^2}\; d\zeta_1 = -\frac{1}{2}\int\limits_{C(\xi_1)+\bar{C}(\xi_1)} f_{\xi_1}(\xi_1,\zeta_1)d\zeta \tag{1.24}$$

Since

$$\int\limits_{C(\xi_1)+\bar{C}(\xi_1)} \frac{\partial}{\partial \nu_1} \ln\sqrt{(\eta_1-f)^2+(\zeta_1-\zeta)^2}\; d\zeta = 2\pi$$

(1.24) becomes

$$S(\xi_1) = \int\limits_{C(\xi_1)} F(\xi_1,\zeta)d\zeta = -\frac{1}{4\pi}\int\limits_{C(\xi_1)} f_{\xi_1}(\xi_1,\zeta)d\zeta = -\frac{1}{4\pi}\frac{d}{d\xi_1}\frac{\bar{A}(\xi_1)}{\sigma^2(\frac{L}{2})^2} \tag{1.25}$$

where $\bar{A}(\xi_1)$ is the area of the section $C(x_1)$.

For $\xi_1 = 1$ is obtained

$$\frac{\partial}{\partial \eta_1}\int_{-d_1}^{o} F(1,\zeta)\ln\sqrt{\eta_1^2+(\zeta_1-\zeta)^2}\; d\xi = -\frac{1}{2} f_{\xi_1}(1,\zeta_1)$$

$$\int_{-d_1}^{o} \frac{\eta_1 F(1,\zeta)}{\eta_1^2+(\zeta_1-\zeta)^2}\, d\zeta = -\frac{1}{2} f_{\xi_1}(1,\zeta_1)$$

for $\eta_1 \to o$:

$$F(1,\zeta_1) = -\frac{1}{2\pi} f_{\xi_1}(1,\zeta_1)$$

and in the same way: (1.26)

$$F(-1,\zeta_1) = -\frac{1}{2\pi} f_{\xi_1}(-1,\zeta_1)$$

By substituting (1.25) and (1.26) in (1.17) through (1.19) the problem is solved.

2. The Wave Resistance

The wave resistance of a slender ship can be calculated by integrating the pressure over the surface of the ship:

$$\bar{R} = \frac{1}{4}\,\sigma^2 L^2 \int_{-1}^{1} d\xi_1 \int_{C(\xi_1)} p\, f_{\xi_1}(\xi_1,\zeta_1)\,d\zeta_1 \tag{2.1}$$

where the pressure p follows from Bernoulli's law, which is for the steady case, deleting higher order terms

$$p = \sigma^2 \rho V^2 \varphi_{\xi_1}(\xi_1,\eta_1,\zeta_1) \tag{2.2}$$

The only contribution to the resistance arises from $\varphi_1(\xi_1,\eta_1,\zeta_1)$. Therefore the dimensionless resistance R becomes

$$R = \frac{\bar{R}}{\frac{1}{2}\rho V^2 L^2} = -\frac{1}{2}\,\sigma^4 \int_{-1}^{1} d\xi_1 \int_{C(\xi_1)} \frac{\partial}{\partial\xi_1}\varphi_1(\xi_1,\eta_1,\zeta_1) f_{\xi_1}(\xi_1,\zeta_1)\,d\zeta \tag{2.3}$$

After one time partial integration:

$$R = -\frac{1}{2}\sigma^4 \Big\{ \int_{C(1)} \varphi_1(1,\eta_1,\zeta_1) f_{\xi_1}(1,\zeta_1)\,d\zeta_1 - \int_{C(-1)} \varphi_1(-1,\eta_1,\zeta_1) f_{\xi_1}(-1,\zeta_1)\,d\zeta_1 + 4\pi \int_{-1}^{1} S_{\xi_1}(\xi_1)\varphi_1(\xi_1)\,d\xi_1 \Big\} \tag{2.4}$$

Substituting (1.17) through (1.19) in (2.4) results in:

$$\begin{aligned} R = -\frac{1}{2}\,\sigma^4 \Big[& \frac{2}{\pi} \int_{-d_1}^{0} f_{\xi_1}(1,\zeta_1)\,d\zeta_1 \int_{-d_1}^{0} f_{\xi}(1,\zeta) \ln\Big\{\frac{\gamma\sigma}{4\beta_0}|\zeta_1+\zeta|\Big\}\,d\zeta \\ & + \frac{2}{\pi} \int_{-d_1}^{0} f_{\xi_1}(-1,\zeta_1)\,d\zeta_1 \int_{-d_1}^{0} f_{\xi}(-1,\zeta) \ln\Big\{\frac{\gamma\sigma}{4\beta_0}|\zeta_1+\zeta|\Big\}\,d\zeta \\ & - 16\pi^2 S(1)S(-1)Y_0\Big(\frac{2}{\beta_0}\Big) - 16\pi^2 S(1) \int_{-1}^{1} S_{\xi}(\xi) Y_0\Big(\frac{1-\xi}{\beta_0}\Big)\,d\xi \end{aligned}$$

$$+ 16\pi^2 S(-1)\int_{-1}^{1} S_\xi(\xi) Y_o\left(\frac{1-\xi}{\beta_o}\right)d\xi + 8\pi^2\int_{-1}^{1} S_{\xi_1}(\xi_1)d\xi_1 \int_{-1}^{1} S_\xi(\xi) Y_o\left(\frac{|\xi_1-\xi|}{\beta_o}\right)d\xi \Bigg] \tag{2.5}$$

This is the same expression as Maruo[2] found and it differs from Vossers's formula only in the first two terms.

For convenience sake the new variables

$$\xi = 2u-1, \quad s = \sigma\zeta, \quad g = \sigma f, \quad d = \sigma d_1, \quad S(\xi) = -\frac{A_u(u)}{8\pi\sigma^2} \tag{2.6}$$

are introduced.

For (2.5) is obtained:

$$R = -\frac{1}{8}\Bigg[\frac{2}{\pi}\int_{-d}^{o} g_u(1,s_1)ds_1 \int_{-d}^{o} g_u(1,s)\ln\{\frac{1}{4}\frac{\gamma}{\beta_o}|s_1+s|\}ds$$

$$+\frac{2}{\pi}\int_{-d}^{o} g_u(-1,s_1)ds_1 \int_{-d}^{o} g_u(-1,s)\ln\{\frac{1}{4}\frac{\gamma}{\beta_o}|s_1+s|\}ds$$

$$- A_u(1)A_u(o)Y_o\left(\frac{2}{\beta_o}\right) - A_u(1)\int_{o}^{1} A_{uu}(u)Y_o\,\{\frac{2}{\beta_o}(1-u)\}du$$

$$+ A_u(o)\int_{o}^{1} A_{uu}(u)Y_o\left(\frac{2}{\beta_o}u\right)du + \frac{1}{2}\int_{o}^{1} A_{u_1u_1}(u_1)du_1$$

$$\int_{o}^{1} A_{uu}(u)Y_o\,\{\frac{2}{\beta_o}|u_1-u|\}du\Bigg] \tag{2.7}$$

It is assumed that $A(u)$ has the Fourier expansion

$$A(u) = \sum_{n=1}^{\infty} a_n \sin n\pi u \tag{2.8}$$

and that $g(u,s)$ is constant fore and aft.

Therefore

$$g_u(1,s) = \frac{\pi}{2d}\sum_{n=1}^{\infty}(-1)^n n\, a_n$$
$$g_u(-1,s) = \frac{\pi}{2d}\sum_{n=1}^{\infty} n\, a_n \tag{2.9}$$

It can be proved by substitutions and partial integration that:

$$\int_0^1 \sin n\pi u\; Y_0\left\{\frac{2}{\beta_0}(1-u)\right\}du = (-1)^{n+1}\int_0^1 \sin n\pi u\; Y_0\left(\frac{2}{\beta_0}u\right)du$$

$$\int_0^1 \sin n\pi u_1\, du_1 \int_0^1 \sin m\pi u\; Y_0\left(\frac{2}{\beta_0}|u_1-u|\right)du$$

$$= \frac{(-1)^{n+m}+1}{\pi(n^2-m^2)}\left\{ n\int_0^1 \sin m\pi u\; Y_0\left(\frac{2}{\beta_0}u\right)du - m\int_0^1 \sin n\pi u\; Y_0\left(\frac{2}{\beta_0}u\right)du\right\}\ ,\ m\neq n$$

$$\int_0^1 \sin n\pi u_1\, du_1 \int_0^1 \sin n\pi u\; Y_0\left\{\frac{2}{\beta_0}|u_1-u|\right\}du$$

$$= \frac{2}{n\pi}\int_0^1 \sin n\pi u\; Y_0\left(\frac{2}{\beta_0}u\right)du + \frac{2}{n\pi\beta_0}\int_0^1 (1-u)\sin n\pi u\; Y_1\left(\frac{2}{\beta_0}u\right)du \tag{2.10}$$

With the notations

$$A_n(\beta_0) = \pi\int_0^1 \sin n\pi u\; Y_0\left(\frac{2}{\beta_0}u\right)du$$
$$B_n(\beta_0) = \pi\int_0^1 (1-u)\sin n\pi u\; Y_1\left(\frac{2}{\beta_0}u\right)du \tag{2.11}$$

the final result becomes:

$$R = \frac{\pi^2}{8}\Bigg[\frac{1}{2\pi}\left(\frac{3}{2} - \ln\frac{\gamma d}{\beta_0}\right)\left\{\left(\sum_{n=1}^{\infty}(-1)^n n a_n\right)^2 + \left(\sum_{n=1}^{\infty} n a_n\right)^2\right\}$$
$$+ Y_0\left(\frac{2}{\beta_0}\right)\left(\sum_{n=1}^{\infty}(-1)^n n a_n\right)\left(\sum_{n=1}^{\infty} n a_n\right) + \left(\sum_{n=1}^{\infty}(-1)^n n a_n\right)\left(\sum_{n=1}^{\infty}(-1)^n n^2 a_n A_n(\beta_0)\right)$$

$$+ \left(\sum_{n=1}^{\infty} n a_n\right)\left(\sum_{n=1}^{\infty} n^2 a_n A_n(\beta_o)\right) + \sum_{n=1}^{\infty}\sum_{m}^{\infty} n^2 m^2 \frac{(-1)^{n+m}+1}{2(n^2-m^2)}(mA_n(\beta_o) - nA_m(\beta_o))a_n a_m$$

$$- \sum_{n=1}^{\infty} n^2 (A_n(\beta_o) + B_n(\beta_o))a_n^2 \Big] \qquad (2.12)$$

To obtain numerical results the series have to be broken off after a finite number of terms N and for a_n has to be taken:

$$a_n = 2\int_o^1 A(u) \sin n\pi u \, du \qquad (2.13)$$

For the determination of the minimum wave resistance the following expression is needed:

$$\frac{\partial R}{\partial a_k} = \sum_{n=1}^{\infty} b_{n,k}\, a_n \qquad (2.14)$$

After some elaboration of the formulas is obtained:

$$b_{2n,2k} = \frac{\pi^2}{8}\Big\{\frac{2}{\pi}\left(\frac{3}{2} - \ln\frac{\gamma d}{\beta_o}\right)(2n)(2k) + 2(2k)(2n)\; Y_o\left(\frac{2}{\beta_o}\right)$$

$$+ 2(2k)(2n)(2nA_{2n} + 2kA_{2k}) + \frac{2(2n)^2(2k)^2}{(2n)^2-(2k)^2}(2kA_{2n} - 2nA_{2k})\Big\}_{n\neq k}$$

$$b_{2n+1,2k+1} = \frac{\pi^2}{8}\Big[\frac{2}{\pi}\left(\frac{3}{2} - \ln\frac{\gamma d}{\beta_o}\right)(2k+1)(2n+1) - 2(2k+1)(2n+1)\; Y_o\left(\frac{2}{\beta_o}\right)$$

$$+ 2(2k+1)(2n+1)\{(2n+1)A_{2n+1} + (2k+1)A_{2k+1}\}$$

$$+ \frac{2(2n+1)^2(2k+1)^2}{(2n+1)^2-(2k+1)^2}\{(2k+1)A_{2n+1} - (2n+1)A_{2k+1}\}\Big]_{n\neq k}$$

$$b_{2k,2k} = \frac{\pi^2}{8}\Big\{\frac{2}{\pi}\left(\frac{3}{2} - \ln\frac{\gamma d}{\beta_o}\right)(2k)^2 + 2(2k)^2\; Y_o\left(\frac{2}{\beta_o}\right) + 4(2k)^3 A_{2k} + 2(2k)^3(A_{2k} + B_{2k})\Big\}$$

$$b_{2k+1,2k+1} = \frac{\pi^2}{8}\Big\{\frac{2}{\pi}\left(\frac{3}{2} - \ln\frac{\gamma d}{\beta_o}\right)(2k+1)^2 - 2(2k+1)^2\; Y_o\left(\frac{2}{\beta_o}\right) + 4(2k+1)^3 A_{2k+1}$$

$$+ 2(2k+1)^3(A_{2k+1} + B_{2k+1})\Big\}$$

$$b_{2n+1,2k} = o \quad ; \quad b_{2n,2k+1} = o \qquad (2.15)$$

The expression for the dimensionless displacement is:

$$D = \frac{\nabla}{L^3} = \frac{1}{4}\int_0^1 A(u)du = \frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}a_n \qquad (2.16)$$

The wave resistance is reduced to a minimum in the condition that the length L and the displacement ∇ are kept constant.

Therefore should be satisfied:

$$\frac{\partial}{\partial a_k}(R+\lambda D) = 0 \quad , k = 1,2,3\ldots \qquad \text{and} \qquad \frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}a_n = D \qquad (2.17)$$

If δ_n is defined by $\delta_n = \frac{a_n}{\lambda}$ it follows that

$$\sum_{n=1}^{\infty} b_{n,k}\delta_n = \frac{(-1)^k-1}{4\pi k} \quad , k = 1,2,3\ldots \qquad \text{and} \qquad \lambda = 4\pi D\left\{\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\delta_n\right\}^{-1} \qquad (2.18)$$

Finally α_n is defined by $a_n = D\alpha_n$, where

$$\alpha_n = 4\pi\delta_n\left\{\sum_{m=1}^{\infty}\frac{1-(-1)^m}{m}\delta_m\right\}^{-1} \qquad (2.19)$$

The δ_n's can be calculated from (2.18) by breaking off the series after a finite number of terms.

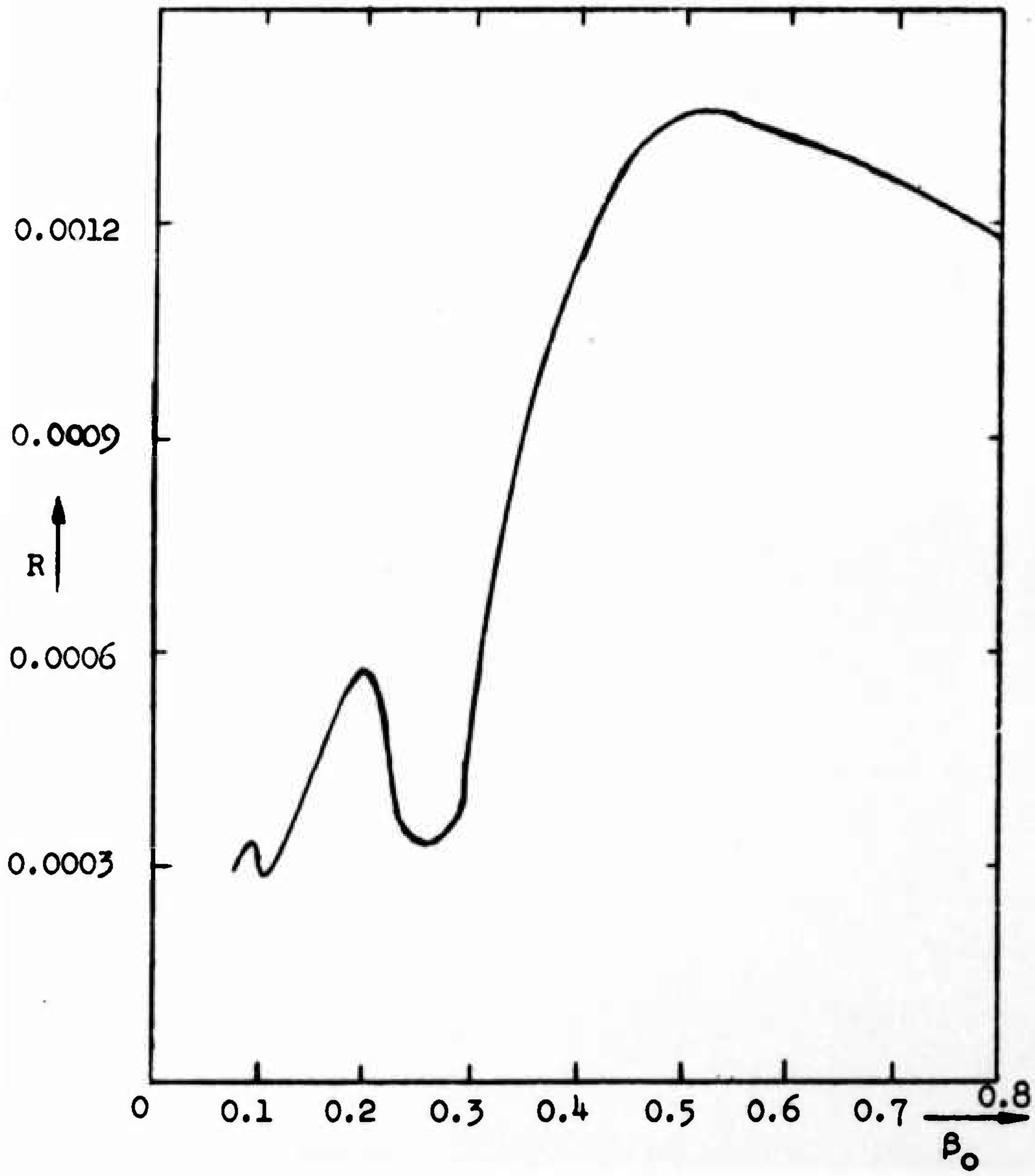
The curve of sectional areas is determined by the formula

$$A(u) = D\sum_{n=1}^{\infty}\alpha_n \sin n\pi u \qquad (2.20)$$

As an example the wave resistance is calculated for the shipform:

$$\frac{B}{L} = 0.1 \quad ; \quad \frac{T}{L} = 0.05 \quad ; \quad A(x) = 0.02 \sin \pi x \quad , 0 \le x \le 1 \; .$$

The values for R are shown in the graph below:



Concerning the problem of minimum wave resistance it may be noted that according to the relations (2.15) and (2.18) the δ_n's with even indices are equal to zero.

Therefore (2.20) becomes $A(u) = D\sum_{n=o}^{\infty} \alpha_{2n+1}\sin(2n+1)\pi u$, which shows that in this case the curve of sectional areas is symmetrical with respect to the midship section.

Part II

THE VELOCITY POTENTIAL FOR THE MOTION OF A SLENDER SHIP OSCILLATING AT ZERO FORWARD SPEED

1. The Case of Small or Moderate Frequency Parameter ξ_L

The same method is followed as in section 1 of part I. Only swaying, yawing, heaving and pitching harmonic motions are considered. The linear displacements of the C G of the ship and the rotations of the ship are assumed to be small of order σ and we put:

$$y_o = \sigma \frac{L}{2} \bar{\eta}_o e^{iwt} \quad , \quad z_o = \sigma \frac{L}{2} \bar{\zeta}_o e^{iwt}$$
$$\psi_o = \sigma \bar{\psi}_o e^{iwt} \quad , \quad \chi_o = \sigma \bar{\chi}_o e^{iwt} \qquad (1.1)$$

If the amplitudes of the rolling and surging motion are of order σ , they will contribute to the slender body theory only in a higher order approximation (cf Vossers[1]). The velocity potential is made dimensionless by writing

$$\phi(x_1,y_1,z_1,t) = \sigma^2 \frac{g}{w} \frac{L}{2} \varphi(\xi_1,\eta_1,\zeta_1) e^{iwt} \qquad (1.2)$$

The definitions of the other variables are given in (I,1.3).

Deleting terms of higher order the boundary condition on the hull becomes

$$\frac{\partial\varphi}{\partial\nu_1} = i\xi_L\{\bar{\eta}_o+\bar{\chi}_o\xi_1+(\bar{\zeta}_o-\bar{\psi}_o\xi_1)f_{\zeta_1}(\xi_1,\zeta_1)\} \ , \ \xi_L = \frac{\omega^2 L}{2g} \qquad (1.3)$$

The potential $\varphi_1(\xi_1,\eta_1,\zeta_1)$ is written as

$$\varphi(\xi_1,\eta_1,\zeta_1) = \int_{-1}^{1} d\xi \int_{C(\xi)} F(\xi,\zeta)\, G(\xi,\eta,\zeta;\ \xi_1,\eta_1,\zeta_1) d\zeta \qquad (1.4)$$

On account of the result (A-5) of the Appendix, may be written:

$$G(\xi,\eta,\zeta;\ \xi_1,\eta_1,\zeta_1) = \frac{1}{\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-f)+\sigma^2(\zeta_1-\zeta)^2}}$$
$$+ \frac{1}{\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-f)^2+\sigma^2(\zeta_1+\zeta)^2}}$$

$$+ 2\pi i\xi_L e^{\sigma\xi_L(\zeta_1+\zeta)} H_o^{(1)}(\xi_L \sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-f)^2})$$

$$- 2\xi_L \int_0^\infty \frac{e^{-\xi_L\lambda}\, d\lambda}{\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-f)^2+\{\sigma(\zeta_1+\zeta)+\lambda\}^2}} \qquad (1.5)$$

Developing the last two terms with respect to σ, yields:

$$2\pi i\ H_o^{(1)}(\xi_L \sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-f)^2}) - 2\xi_L \int_0^\infty \frac{e^{-\xi_L\tau}\, d\tau}{\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-f)^2+\tau^2}}$$

$$+ 2\xi_L \int_0^{\sigma(\zeta_1+\zeta)} \frac{e^{-\xi_L\tau}\, d\tau}{\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-f)^2+\tau^2}} + O(\sigma)$$

$$(1.6)$$

With the relation

$$\int_0^\infty \frac{e^{-\xi_L\tau}\, d\tau}{\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-f)^2+\tau^2}} = \frac{2}{\pi}\int_0^\infty K_o(p\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-f)^2})dp \int_0^\infty e^{-\xi_L\tau}\cos p\tau\, d\tau$$

$$= \frac{2\xi_L}{\pi}\int_0^\infty \frac{K_o(p\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-f)^2})}{p^2+\xi_L^2}\, dp$$

$$= \frac{\pi}{2}\{ H_o(\xi_L\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-f)^2})$$

$$- Y_o(\xi_L\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-f)^2})\}$$

(1.6) becomes:

$$2\pi i\xi_L\ H_o^{(1)}(\xi_L\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-f)^2}) - \pi\xi_L\{H_o(\xi_L\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-f)^2})$$

$$- Y_o(\xi_L\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-f)^2})\} + 2\xi_L\int_0^{\sigma(\zeta_1+\zeta)} \frac{d\tau}{\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-f)^2+\tau^2}} + O(\sigma)$$

With $\int_{C(\xi)} F(\xi,\zeta)d\zeta = S(\xi)$ and assuming ξ_L to be $O(1)$ or smaller, the potential for $|\xi_1| \neq 1$ can, deleting terms of higher order than zero, be written as

$$\varphi(\xi_1,\eta_1,\zeta) = \int_{-1}^{1} d\xi \int_{C(\xi)+\bar{C}(\xi)} \frac{F(\xi,\zeta)d\zeta}{\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-f)^2+\sigma^2(\zeta_1-\zeta)^2}}$$
$$+ \pi\xi_L \int_{-1}^{1} S(\xi)\{2iH_o^{(1)}(\xi_L|\xi_1-\xi|)-H_o(\xi_L|\xi_1-\xi|)+Y_o(\xi_L|\xi_1-\xi|)\} \tag{1.7}$$

Expansion of the first term results in (cf Ward[4]):

$$\varphi(\xi_1,\eta_1,\zeta_1) = -2\int_{C(\xi_1)+\bar{C}(\xi_1)} F(\xi_1,\zeta)\ln\sigma\sqrt{(\eta_1-f)^2+(\zeta_1-\zeta)^2}\, d\zeta + 2S(-1)\ln 2(1+\xi_1)$$
$$+2S(1)\ln 2(1-\xi_1) + 2\int_{-1}^{\xi_1} S_\xi(\xi)\ln 2|\xi_1-\xi|d\xi - 2\int_{\xi_1}^{1} S_\xi(\xi)\ln 2|\xi_1-\xi|d\xi$$
$$+\pi\xi_L\int_{-1}^{1} S(\xi)\{2iH_o^{(1)}(\xi_L|\xi_1-\xi|)-H_o(\xi_L|\xi_1-\xi|)+Y_o(\xi_L|\xi_1-\xi|)\}d\xi \tag{1.8}$$

The integral equation for $F(\xi,\zeta)$ is obtained from the boundary condition (1.3)

$$\frac{\partial}{\partial\nu_1}\int_{C(\xi_1)+\bar{C}(\xi_1)} F(\xi_1,\zeta)\ln\sigma\sqrt{(\eta_1-f)^2+(\zeta_1-\zeta)^2}d\zeta = -\frac{1}{2}i\xi_L\{\bar{\eta}_o+\bar{X}_o\xi_1+(\bar{\zeta}_o-\bar{\psi}_o\xi_1)f_{\zeta_1}(\xi_1,\zeta_1)\}$$

from which follows in the same way as the derivation of formula (1.25) of part I:

$$S(\xi_1) = -\frac{i}{4\pi}\xi_L\{(\bar{\eta}_o+\bar{X}_o\xi_1)\Gamma(\xi_1)+(\bar{\zeta}_o-\bar{\psi}_o\xi_1)b(\xi_1)\} \tag{1.9}$$

where $\Gamma(\xi_1) = \int_C d\zeta$ is the length of the contour $C(\xi_1)$ and $b(\xi_1) = f(\xi_1,0)$ is the beam at the point ξ_1.

For a slender body of revolution formula (1.8) becomes the same as Ursell found in Reference 3.

2. The Case of Large Frequency Parameter ξ_L

The frequency parameter is assumed to be $\xi_L = \frac{\xi_B}{\sigma}$, where

$$\xi_B = \frac{\omega^2 B}{2g} = O(1) \tag{2.1}$$

The definition of the potential is the same as in the preceding section. The displacements are chosen as:

$$y_o = \sigma^2 \frac{L}{2} \bar{\eta}_o e^{iwt} \quad , \quad z_o = \sigma^2 \frac{L}{2} \bar{\zeta}_o e^{iwt}$$

$$\psi_o = \sigma^2 \bar{\psi}_o e^{iwt} \quad , \quad \chi_o = \sigma^2 \bar{\chi}_o e^{iwt}$$

For Green's function is taken the form (A-6):

$$\begin{aligned} G(\xi,\eta,\zeta;\ \xi_1,\eta_1,\zeta_1) &= \frac{1}{\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-f)^2+\sigma^2(\zeta_1-\zeta)^2}} \\ &- \frac{1}{\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-f)^2+\sigma^2(\zeta_1+\zeta)^2}} \\ &- 2\sigma\xi_L \int_0^\infty e^{-\sigma\xi_L\lambda} \Big\{ \frac{1}{\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-f)^2+\sigma^2(\zeta_1+\zeta+\lambda)^2}} \\ &- \frac{1}{\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-f)^2+\sigma^2(\zeta_1+\zeta)^2}} \Big\} d\lambda \\ &+ 2\pi i \xi_L e^{\sigma\xi_L(\zeta_1+\zeta)} H_o^{(1)}\left(\xi_L \sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-f)^2}\right) \end{aligned} \tag{2.2}$$

Substituting (2.2) in (1.4):

$$\begin{aligned} \varphi(\xi_1,\eta_1,\zeta_1) &= \int_{-1}^{1} d\xi \int_{C(\xi)} F(\xi,\zeta) \Big\{ \frac{1}{\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-f)^2+\sigma^2(\zeta_1-\zeta)^2}} \\ &- \frac{1}{\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-f)^2+\sigma^2(\zeta_1+\zeta)^2}} \Big\} d\zeta \\ &- 2\sigma\xi_L \int_0^\infty e^{-\sigma\xi_L\lambda} d\lambda \int_{-1}^{1} d\xi \int_{C(\xi)} F(\xi,\zeta) \Big\{ \frac{1}{\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-f)^2+\sigma^2(\zeta_1+\zeta+\lambda)^2}} \end{aligned}$$

$$- \frac{1}{\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-f)^2+\sigma^2(\zeta_1+\zeta)^2}}\Big\} d\zeta$$

$$+ 2\pi i \xi_L \int_{-1}^{1} d\xi \int_{C(\xi)} F(\xi,\zeta) e^{\sigma\xi_L(\zeta_1+\zeta)} H_0^{(1)}\left(\frac{\xi_B}{\sigma}\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-f)^2}\right) d\zeta \tag{2.3}$$

After one time partial integration with respect to ξ of the first two integrals, using the asymptotic behavior of the Hankel function and neglecting terms of higher order, the potential becomes:

$$\varphi(\xi_1,\eta_1,\zeta_1) = -2\int_{C(\xi_1)} F(\xi_1,\zeta)\left\{\ln\sigma\sqrt{(\eta_1-f)^2+(\zeta_1-\zeta)^2}+\ln\sigma\sqrt{(\eta_1-f)^2+(\zeta_1+\zeta)^2}\right\} d\zeta$$

$$+ 4\sigma\xi_L \int_0^{\infty} e^{-\sigma\xi_L\lambda} d\lambda \int_{C(\xi_1)} F(\xi_1,\zeta) \ln\sigma\sqrt{(\eta_1-f)^2+(\zeta_1+\zeta+\lambda)^2}\, d\zeta$$

$$+ 2\sqrt{2\pi}\,\frac{\xi_B}{\sigma}\, i \int_{-1}^{1} d\xi \int_{C(\xi)} F(\xi,\zeta) e^{\xi_B(\zeta_1+\zeta)} \left\{\frac{\sigma}{\xi_B\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-f)^2}}\right\}^{\frac{1}{2}}$$

$$e^{i\frac{\xi_B}{\sigma}\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-f)^2} - \frac{\pi}{4}i}\, d\zeta \tag{2.4}$$

By applying the method of stationary phase on the last integral is obtained:

$$2\sqrt{2\pi}\,\frac{\xi_B}{\sigma}\, i \int_{-1}^{1} d\xi \int_{C(\xi)} F(\xi,\zeta) e^{\xi_B(\zeta_1+\zeta)} \left\{\frac{\sigma}{\xi_B\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-f)^2}}\right\}^{\frac{1}{2}}$$

$$e^{i\frac{\xi_B}{\sigma}\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-f)^2} - \frac{\pi}{4}i}\, d\zeta = 4\pi i \int_{C(\xi_1)} F(\xi_1,\zeta) e^{\xi_B(\zeta_1+\zeta)+i\xi_B(\eta_1-f)}\, d\zeta \tag{2.5}$$

Substitution of (2.5) in (2.4) yields the final result

$$\varphi(\xi_1,\eta_1,\zeta_1) = \int_{C(\xi_1)} F(\xi_1,\zeta) \Big[-2\ln\sqrt{(\eta_1-f)^2+(\zeta_1-\zeta)^2} - 2\ln\sqrt{(\eta_1-f)^2+(\zeta_1+\zeta)^2}$$

$$+ 4\xi_B \int_0^{\infty} e^{-\xi_B\lambda} \ln\sqrt{(\eta_1-f)^2+(\zeta_1+\zeta+\lambda)^2}\, d\lambda + 4\pi i e^{\xi_B\{(\zeta_1+\zeta)+i(\eta_1-f)\}}\Big] d\zeta \tag{2.6}$$

This zero order term is exactly the same as the potential for the motion of an infinite cylinder with section $C(\xi_1)$, which follows from (A-7).

From the boundary condition

$$\frac{\partial\varphi}{\partial\nu_1} = i\xi_B\left\{\bar{\eta}_o+\bar{\chi}_o\xi_1 + (\bar{\zeta}_o-\bar{\psi}_o\xi_1)f_{\zeta_1}(\xi_1,\zeta_1)\right\}$$

it follows that

$$2\int_{C(\xi_1)} F(\xi_1,\zeta)\Bigg[-\frac{\eta_1-f+(\zeta_1-\zeta)f_{\xi_1}(\xi_1,\zeta_1)}{(\eta_1-f)^2+(\zeta_1-\zeta)^2}-\frac{\eta_1-f+(\zeta_1+\zeta)f_{\xi_1}(\xi_1,\zeta_1)}{(\eta_1-f)^2+(\zeta_1+\zeta)^2}$$

$$+2\xi_B\int_0^\infty e^{-\xi_B\lambda}\frac{\eta_1-f+(\zeta_1+\zeta+\lambda)f_{\xi_1}(\xi_1,\zeta_1)}{(\eta_1-f)^2+(\zeta_1+\zeta+\lambda)^2}\,d\lambda+2\pi i\xi_B\left\{i+f_{\zeta_1}(\xi_1\zeta_1)\right\}$$

$$\exp\left[\xi_B\left\{(\zeta_1+\zeta)+i(\eta_1-f)\right\}\right]\Bigg]d\zeta = i\xi_B\left\{\bar{\eta}_o+\bar{\chi}_o\xi_1+(\bar{\zeta}_o-\bar{\psi}_o\xi_1)f_{\xi_1}(\xi_1,\zeta_1)\right\} \tag{2.7}$$

This integral equation determines the source distribution $F(\xi_1\zeta)$.

APPENDIX

GREEN'S FUNCTION FOR THE FREE SURFACE CONDITION

A Green's function for the free surface condition can be derived by different methods. See f.i. Havelock,[5] Timman and Vossers,[6] Peters and Stoker.[7]

For the steady case it is usually expressed in the dimensionless form:

$$G(\xi,\eta,\zeta;\ \xi_1,\eta_1,\zeta_1) = \frac{1}{\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-\eta)^2+\sigma^2(\zeta_1-\zeta)^2}} - \frac{1}{\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-\eta)^2+\sigma^2(\zeta_1+\zeta)^2}}$$

$$- \frac{4}{\pi\beta_o}\,\mathcal{R}e\int^{\frac{\pi}{2}} d\Theta \int_{\overline{M}} \frac{e^{q\{\sigma(\zeta_1+\zeta)+i\sigma(\eta_1-\eta)\sin\Theta+i(\xi_1-\xi)\cos\Theta\}}}{q\cos\Theta-\frac{1}{\beta_o}}\,dq \tag{A-1}$$

$\mathcal{R}e$ means the real part and the contour $\overline{M}$ passes the pole in the upper half plane in order to satisfy the condition for $\xi_1 \to \infty$. After introduction of the new parameters

$$\sinh\tau = tg\Theta$$

and

$$p = \frac{q}{\cosh^2\tau}$$

(A-1) becomes:

$$G(\xi,\eta,\zeta;\ \xi_1,\eta_1,\zeta_1) = \frac{1}{\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-\eta)^2+\sigma^2(\zeta_1-\zeta)^2}} - \frac{1}{\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-\eta)^2+\sigma^2(\zeta_1+\zeta)^2}}$$

$$- \frac{2}{\pi\beta_o}\,\mathcal{R}e\int_{-\infty}^{\infty}\cosh\tau\, d\tau \int_{\overline{M}} \frac{e^{p\{\sigma(\zeta_1+\zeta)\cosh^2\tau+i\sigma(\eta_1-\eta)\sinh\tau\cosh\tau+i(\xi_1-\xi)\cosh\tau\}}}{p-\frac{1}{\beta_o}}\,dp$$

with M :

$p = \frac{1}{\beta_o}$

or

$$G(\xi,\eta,\zeta;\xi_1,\eta_1,\zeta_1) = \frac{1}{\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-\eta)^2+\sigma^2(\zeta_1-\zeta)^2}} + \frac{1}{\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-\eta)^2+\sigma^2(\zeta_1+\zeta)^2}}$$

$$- \frac{2}{\pi} \mathcal{R}e \int_{-\infty}^{\infty} \cosh\tau d\tau \int_M \frac{p\, e^{p\{\sigma(\zeta_1+\zeta)\cosh^2\tau + i\sigma(\eta_1-\eta)\sinh\tau\cosh\tau + i(\xi_1-\xi)\cosh\tau\}}}{p-\frac{1}{\beta_o}} dp \tag{A-2}$$

where the representation

$$\frac{1}{\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-\eta)^2+\sigma^2(\zeta_1+\zeta)^2}} = \frac{2}{\pi}\int_0^{\infty} K_o(q\sigma\sqrt{(\zeta_1+\zeta)^2+(\eta_1-\eta)^2})\cos q(\xi_1-\xi)dq$$

$$= \frac{\mathcal{R}e}{\pi}\int_{-\infty}^{\infty}\cosh\tau d\tau \int_0^{\infty} e^{p\{\sigma(\zeta_1+\zeta)\cosh^2\tau+i\sigma(\eta_1-\eta)\sinh\tau\cosh\tau+i(\xi_1-\xi)\cosh\tau\}} dp$$

is used.

The third term in (A-2) can be written:

$$\mathcal{J} = -i\int_{-\infty}^{\infty} p\cosh\tau\, e^{p\{\sigma(\zeta_1+\zeta)\cosh^2\tau+i\sigma(\eta_1-\eta)\sinh\tau\cosh\tau+i(\xi_1-\xi)\cosh\tau\}} d\tau$$

$$= \frac{\partial}{\partial\xi}\int_{-\infty}^{\infty} e^{p\{\sigma(\zeta_1+\zeta)\cosh^2\tau+i\sigma(\eta_1-\eta)\sinh\tau\cosh\tau+i(\xi_1-\xi)\cosh\tau\}} d\tau$$

which can be transformed into

$$\mathcal{J} = e^{\frac{1}{2}p\sigma(\zeta_1+\zeta)} \frac{\partial}{\partial\xi}\Big\{\int_0^{\infty} e^{-ap\sigma\cosh(2\tau+2i\tau_o)+ip(\xi_1-\xi)\cosh\tau} d\tau$$

$$+ \int_0^{\infty} e^{-ap\sigma\cosh(2\tau-2i\tau_o)+ip(\xi_1-\xi)\cosh\tau} d\tau \Big\}$$

where

$$a = \frac{1}{2}\sqrt{(\zeta_1+\zeta)^2+(\eta_1-\eta)^2}$$

$$\tau_o = \frac{1}{2}\ \text{arctg}\ \frac{\eta_1-\eta}{\zeta_1+\zeta} \quad , \ -\frac{\pi}{2} \leq 2\tau_o \leq \frac{\pi}{2}$$

After substitution of new variables

$$\mathcal{J} = e^{\frac{1}{2}p\sigma(\zeta_1+\zeta)} \frac{\partial}{\partial\xi}\left\{\int_0^\infty e^{-p\sigma a\cosh 2z+ip(\xi_1-\xi)\cosh(2-i\tau_o)}\,dz + \int_0^\infty e^{-p\sigma a\cosh 2z+ip(\xi_1-\xi)\cosh(2+i\tau_o)}\,dz\right\}$$

$$= e^{\frac{1}{2}p\sigma(\zeta_1+\zeta)} \frac{\partial}{\partial\xi}\left\{\int_{i\tau_o}^{i\tau_o+\infty} e^{-p\sigma a\cosh 2z+ip(\xi_1-\xi)\cosh(2-i\tau_o)}\,dz + \int_{-i\tau_o}^{-i\tau_o+\infty} e^{-pa\cosh 2z+ip(\xi_1-\xi)\cosh(2+i\tau_o)}\,dz\right\}$$

$$= e^{\frac{1}{2}p\sigma(\zeta_1+\zeta)} \frac{\partial}{\partial\xi}\left\{\int_0^\infty e^{-p\sigma a\cosh 2\tau+ip(\xi_1-\xi)\cosh(\tau-i\tau_o)}\,d\tau + \int_0^\infty e^{-\tau\sigma a\cosh 2\tau+ip(\xi_1-\xi)\cosh(\tau+i\tau_o)}\,d\tau\right\}$$

$$= e^{\frac{1}{2}p\sigma(\zeta_1+\zeta)} \frac{\partial}{\partial\xi}\int_{-\infty}^\infty e^{-p\sigma a\cosh 2\tau+pb(\xi_1-\xi)\sinh\tau+ipc(\xi_1-\xi)\cosh\tau}\,d\tau$$

with $\quad b = -\left\{\frac{2a-(\zeta_1+\zeta)}{4a}\right\}^{\frac{1}{2}} \text{sgn}(\eta_1-\eta) \quad ; \quad c = \left\{\frac{2a+(\zeta_1+\zeta)}{4a}\right\}^{\frac{1}{2}}$

Finally (A-2) becomes:

$$G(\xi,\eta,\zeta;\xi_1,\eta_1,\zeta_1) = \frac{1}{\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-\eta)^2+\sigma^2(\zeta_1+\zeta)^2}} + \frac{1}{\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-\eta)^2+\sigma^2(\zeta_1+\zeta)^2}}$$

$$+ \frac{2}{\pi} \mathcal{R}e\, i \frac{\partial}{\partial\xi} \int_{-\infty}^{\infty} d\tau \int_M \frac{\beta_o e^{\frac{1}{2}p\sigma(\zeta_1+\zeta)-p\sigma a\cosh 2\tau+pb(\xi_1-\xi)\sinh\tau+ipc(\xi_1-\xi)\cosh\tau}\, dp}{\beta_o p-1} \tag{A-3}$$

For the unsteady case of zero headway the Green's function is, see Peters and Stoker[7]

$$G(\xi,\eta,\zeta;\xi_1,\eta_1,\zeta_1) = \frac{1}{\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-\eta)^2+\sigma^2(\zeta_1-\zeta)^2}} + \frac{1}{\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-\eta)^2+\sigma^2(\zeta_1+\zeta)^2}}$$

$$+ 2\xi_L \int_M e^{p\sigma(\zeta_1+\zeta)} \frac{\mathcal{J}_o(p\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-\eta)^2})}{p-\xi_L}\, dp \tag{A-4}$$

with M :

$p = \xi_L$

The third term can be written in the form:

$$2\xi_L \int_M e^{p\sigma(\zeta_1+\zeta)} \frac{\mathcal{J}_o(p\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-\eta)^2}\,)}{p-\xi_L}\, dp$$

$$= 2\pi i \xi_L e^{\sigma\xi_L(\zeta_1+\zeta)} H_o^{(1)}(\xi_L\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-\eta)^2})$$

$$+ 2\xi_L \mathcal{R}e \int_M e^{p\sigma(\zeta_1+\zeta)} \frac{H_o^{(1)}(p\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-\eta)^2})}{p-\xi_L}\, dp$$

$$= 2\pi i \xi_L e^{\sigma\xi_L(\zeta_1+\zeta)} H_o^{(1)}(\xi_L\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-\eta)^2})$$

$$+ 2\xi_L \mathcal{R}e \int_0^{\infty} e^{-\xi_L\lambda} d\lambda \int_0^{\infty} e^{\{\sigma(\zeta_1+\zeta)+\lambda\} ip} K_o(p\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-\eta)^2})\, dp$$

$$= 2\pi i \xi_L e^{\sigma\xi_L(\zeta_1+\zeta)} H_o^{(1)}\left(\xi_L\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-\eta)^2}\right)$$

$$- 2\xi_L \int_0^\infty \frac{e^{-\xi_L\lambda}\, d\lambda}{\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-\eta)^2+\{\sigma(\zeta_1+\zeta)+\lambda\}^2}}$$

Then (A-4) becomes:

$$G(\xi,\eta,\zeta;\xi_1,\eta_1,\zeta_1) = \frac{1}{\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-\eta)^2+\sigma^2(\zeta_1-\zeta)^2}} + \frac{1}{\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-\eta)^2+\sigma^2(\zeta_1+\zeta)^2}}$$

$$+ 2\pi i \xi_L e^{\sigma\xi_L(\zeta_1+\zeta)} H_o^{(1)}\left(\xi_L\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-\eta)^2}\right)$$

$$- 2\xi_L \int_0^\infty \frac{e^{-\xi_L\lambda}\, d\lambda}{\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-\eta)^2+\{\sigma(\zeta_1+\zeta)+\lambda\}^2}} \qquad \text{(A-5)}$$

or

$$G(\xi,\eta,\zeta;\xi_1,\eta_1,\zeta_1) = \frac{1}{\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-\eta)^2+\sigma^2(\zeta_1-\zeta)^2}} - \frac{1}{\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-\eta)^2+\sigma^2(\zeta_1+\zeta)^2}}$$

$$+ 2\pi i \xi_L e^{\sigma\xi_L(\zeta_1+\zeta)} H_o^{(1)}\left(\xi_L\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-\eta)^2}\right)$$

$$- 2\sigma\xi_L \int_0^\infty e^{-\sigma\xi_L\lambda}\left\{\frac{1}{\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-\eta)^2+\sigma^2(\zeta_1+\zeta+\lambda)^2}} - \frac{1}{\sqrt{(\xi_1-\xi)^2+\sigma^2(\eta_1-\eta)^2+\sigma^2(\zeta_1+\zeta)^2}}\right\} d\lambda \qquad \text{(A-6)}$$

The Green's function for the two dimensional case can be found by integration of (A-6):

$$g(\eta,\zeta;\eta_1,\zeta_1) = \int_{-\infty}^{\infty} G(\xi,\eta,\zeta;\xi_1,\eta_1,\zeta_1)\, d\xi$$

resulting in

$$g(\eta,\zeta;\eta_1,\zeta_1) = -2\ln\sqrt{(\eta_1-\eta)^2+(\zeta_1-\zeta)^2} \quad -2\ln\sqrt{(\eta_1-\eta)^2+(\zeta_1+\zeta)^2}$$

$$+ 4\sigma\xi_L \int_0^\infty e^{-\sigma\xi_L\lambda} \ln\sqrt{(\eta_1-\eta)^2+(\zeta_1+\zeta+\lambda)^2}\, d\lambda \quad + 4\pi i e^{\sigma\xi_L\{(\zeta_1+\zeta)+i(\eta_1-\eta)\}} \tag{A-7}$$

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DISCUSSIONS

by J. N. Newman

Mr. Joosen's paper helps to anser some of the questions which have stemmed from a close examination of Vossers' pioneering effors, and as such is most welcome. It should be emphasized however that in one fundamental respect the present paper differs from Vossers, which is based throughout on the construction of the velocity potential by means of Green's theorem. Mr. Joosen, on the other hand, uses a source distribution of unknown strength, ultimately determining this from the boundary condition on the body. I should first like to ask, for academic interest, why this alternative was chosen. Secondly I would question the way the boundary condition is satisfied at the ends, for as shown in Equations (1.18) and (1.19), φ_1 depends on the transverse coordinates η_1 and ζ_1. It would seem, therefore, that the end boundary conditions (1.26) should contain a contribution from the wave potential φ_1. Finally I wish to comment on the Green's function (1.8) which is constructed in an ingeneous manner. The difficulty of breaking the hull integration into two additional parts depending on the sign of $\eta_1 - f$ is well known, but I wonder if it is in fact possible to get around this complication? With regard to (1.8), if we consider $\xi_1 - \xi > 0$, the integral over p from 0 to $+i\infty$ is clearly convergent, but is this so along the quadrant, or for that matter along the real axis, for the term

$$pb\ (\xi_1 - \xi)\ \sin h\ \tau$$

will tend to either $+\infty$ or $-\infty$ depending on the sign of $(\eta_1 - f)$.

If I can be excused for a brief discussion of the oscillatory problem in Part II, it is especially welcome to see in the same paper a comparison of the two cases where $\omega^2 L/g$ or $\omega^2 B/g$ are of order one. In the first case I have obtained similar results, starting with Green's theorem, in a paper which will appear in the Journal of Fluid Mechanics. The first case would seem more practical, since intuitively one would think that the important wavelengths are of the same order of magnitude as the ship's length, but the resulting forces are relatively trivial; both the body's own inertial force and the usual damping and "added mass" forces are of higher order than the "Fronde-Krylov" exciting and hydrostatic restoring forces. This situation is, in a sense, even more degenerate than the first order thin ship theory. Nevertheless there has

recently been some rather startling experimental confirmation of the first order slender body theory, with $\omega^2 L/g = O(1)$, for a conventional ship hull at zero speed in head waves. This comparison is, I believe, to be presented by Dr. Cummins at the I.T.T.C. meetings next month. Although it is a slight digression from wave resistance, I would welcome Mr. Joosen's comments on the comparative practical value of the two different cases which he has considered.

by H. Maruo

See discussion by Maruo in previous paper.

AUTHOR'S REPLY

I wish to thank Professor Maruo for his contribution to the discussion of my paper. I wonder whether it is worthwhile to perform such a considerable amount of work to obtain the second order term. In my opinion it is better to consider a direct, numerical approach to the problem. In reply to Dr. Newman I would like to say that I had no special reasons to make use of source distributions rather than Green's theorem.

Perhaps it would be more straight forward to start from Green's theorem, on the other hand this would involve the evaluation of two integrals instead of one.

From Equations (1.18) and (1.19) it is clear that the part of φ_1 , that depends on η_1, and ζ_1, represents a two dimensional source distribution on the line segment $\eta = 0$, $\zeta > 0$. Therefore the induced velocity in η_1-direction equals zero on the line segment $\eta_1 = 0$, $\zeta_1 < 0$.

With regard to the difficulty in Equation (1.8) I would remark that the concept of generalized functions is used there. It is for instance possible to multiply the integrand by a factor $\exp\{\varepsilon(i-1)p\sqrt{p}\}$ in which ε is a small positive quantity, that tends to zero in the final result. Then it is evident that the integrals over p remain convergent in the whole quadrant.

Considering the oscillatory problem for the case

$$\frac{\omega^2 L}{g} = \frac{2\pi L}{\lambda} = 1$$

it can be concluded that the corresponding wave length λ is a multiple of the ship length. If for the second case is taken $\frac{\omega^2 L}{g} = \frac{1}{\sigma}$ and assuming $0.1 < \sigma < 0.2$ the value of $\frac{\lambda}{L}$ lies somewhere between 0.6 and 1.2. Therefore this case corresponds to the situation, where λ is of the same order of magnitude as the ship length.

LINEAR THEORIES FOR THE MOTION OF SHIPS

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LINEAR THEORIES FOR THE MOTION OF SHIPS*

1. Introductory Remarks

In this talk I will not present any new results, but rather review some previous work by A. S. Peters and myself,[1] and of others, perhaps with some new emphasis and interpretations.

I speak as a mathematician who has the object of treating the motion of a ship in a seaway on the basis of the hydrodynamical theory of a perfect incompressible gravitating fluid. In doing so, the ship is regarded as a floating rigid body in motion under the action of the pressure forces exerted by the water on its hull, and of a prescribed propulsive force. No a priori assumptions concerning the coupling of the motion of the ship and of the water are to be made, nor of the coupling between the various degrees of freedom of the motion of the ship - rather, these effects are to be calculated as part of the solution of the problem. If this problem could be solved for relevant ranges of the parameters involved it would without question (at least that is an article of faith with me) be helpful in the end in practice, even though such important physical effects as those due to viscosity, a turbulent wake, etc. are ignored. Unfortunately even the problem for perfect fluids is beyond the power of modern mathematical analysis unless further simplifications are made, since it is nonlinear and a free boundary occurs. In such situations a time-honored procedure has often led to useful approximate theories, and that is a linearization with respect to some dimensionless parameter that is very small (or, perhaps on occasion, very large). This means, in effect, that not one problem but a whole sequence of problems depending on the parameter is considered, and the solution is to be approximated in a neighborhood of the zero value of the parameter. Commonly, this is done by assuming that the solution can be developed in integral powers of the parameter, with coefficients that depend on the space variables and the time, and that a few terms in the series will suffice to give a reasonably

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[1]The Motion of a Ship, as a Floating Rigid Body, in a Seaway: Comm. Pure Appl. Math., Vol. X, 1957. All papers to which references are made are listed in the bibliography of this paper and will not be given here.

accurate solution. That is, in brief, the procedure to be outlined and discussed here.[2]

The linear theories to be described here are all based on the assumption that the ship can be regarded as a slender disk. The motivation for this is that the forward speed be a thin disk so oriented with respect to its forward velocity that it can slice through the water without causing disturbances of finite amplitude, since the free surface condition could not reasonably be linearized unless that were so. Thus the linearizations are carried out with respect to a small slenderness parameter. This is what was done by Michell in his classic paper on the wave resistance of ships, though he does not put the matter in this way. The original purpose of A. S. Peters and me was to generalize the Michell theory to the maximum extent possible in allowing the ship to be a floating body

[2]There is some evidence, at least, that such a procedure may be reasonable from the mathematical point of view for surface wave problems. Levi-Civita, in a classic paper, proved rigorously the existence of plane progressing waves of finite amplitude in the framework of the full nonlinear theory by showing that the perturbation series in the amplitude as small parameter do indeed converge to the solution. That the series would converge in the present cases is doubtful unless special precautions were to be taken at the water line of the ship, but the author has no doubt that the perturbation procedure yields a valid approximation in at least an asymptotic sense. In this connection, it is perhaps worth observing here that the classical procedure of Rayleigh and others for the treatment of the solitary wave (which starts with a nonlinear lowest order approximation) has been shown by Friedrichs and Hyers to be the correct lowest order term of an asymptotic perturbation procedure that is valid in the neighborhood of the critical speed for the given depth of channel; here the parameter with respect to which the approximation is made is

$$\frac{v - v_{crit}}{\sqrt{gh}},$$

with v the propagation speed of the wave.

(whereas Michell regarded it as, in effect, held fixed in space with the water streaming past). It was, of course, necessary to make the hypothesis that the oscillations of the ship should have small amplitudes; this was done in assuming that the quantities characterizing these motions could also be developed in powers of the slenderness parameter. After carrying out the formal developments in order to obtain the generalization of Michell's theory it was seen that various conclusions of general interest could be drawn simply from an inspection of the resulting theory. For example, it turned out that certain of the modes of oscillation of the ship are not damped at least in lowest order - while others are. In fact, the rolling, yawing, and swaying oscillations are damped, while all others, i.e. the surging, pitching, and heaving oscillations are not damped to first order. This conclusion from the generalized Michell's theory has not been received with happiness, since it seems to be true that for actual ships the pitching and heaving modes are damped to about the same extent as the other modes. However, that this result would be inevitable in such a theory is easily seen upon a little reflection. It comes about because of the way in which the slenderness parameter is defined for the generalization of the Michell theory, i.e. from the assumption that the ship is a thin disk the plane of symmetry of which is the fore-and-aft vertical midsection of the ship. In the absence of friction, the only mechanism available for damping is the creation of surface waves which carry off energy to infinity. Clearly, for a ship of Michell's type an oscillation of infinitesimal amplitude in pitching or heaving (and surging) would create surface waves having amplitudes of second order at least, while the other modes in which the flat surface of the disk is pressed against the water can cause surface waves of first order when these particular oscillations also have amplitudes of first order.

Once these observations were made, Peters and I saw that other ways of linearization could be introduced in such a fashion that the pitching and heaving oscillations would be damped to first order. One way to achieve that is to regard the ship as slender with respect to the horizontal plane at the water line - and thus to regard it as a limit case in which the ship is planing over the water (cf. Figure 1). Such a procedure recommends itself as quite possibly better than that involved in the generalization of Michell's theory since it means that it is now the draft-length ratio of the ship that is regarded as small while in the former case it is the beam-length ratio that is small - and it is a fact that for most ships the draft-length ratio is noticeably the smaller of the two ratios. The same kind of thought experiment as was carried out above for the Michell-type ship indicates clearly that the planing type ship would yield a

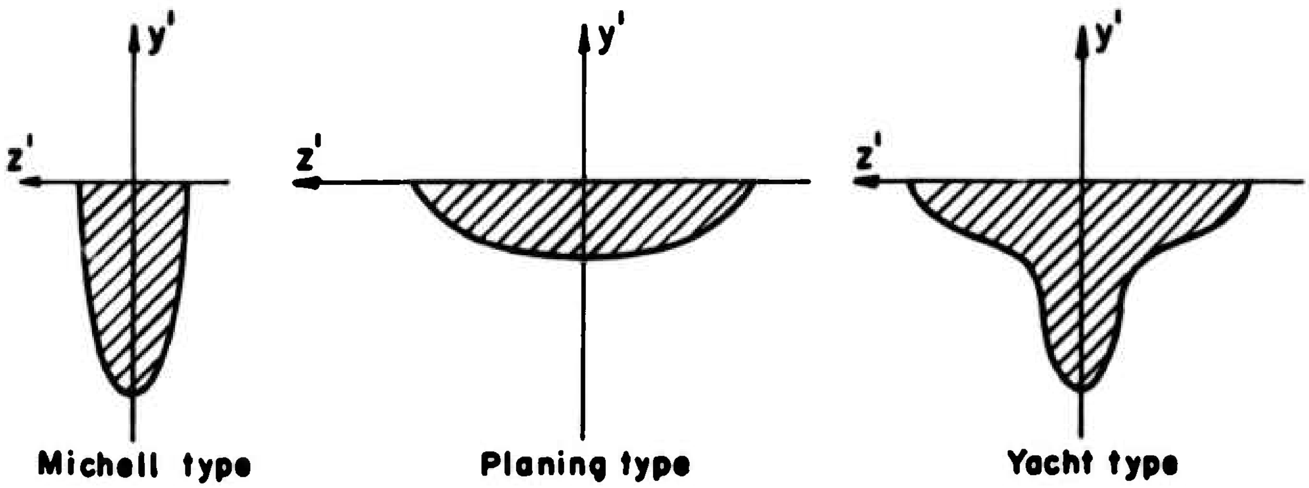
theory for which the heaving, pitching, and rolling oscillations are damped, while the yawing, swaying, and surging oscillations are not damped (to first order). Finally, both types of hulls could be combined into what is called in Figure 1 a yacht-type hull; with this shape of hull all modes of oscillation, except the surging oscillation, would be damped. In the following sections a brief outline of the results of developments in these cases will be given.[3]

2. The Generalization of Michell's Theory

Figures 2 and 3 indicate the coordinate systems employed in what follows. The system X, Y, Z is fixed in space with the X, Z-plane in the undisturbed free surface, and the Y-axis vertically upward. The x, y, z coordinate system is taken with the y-axis vertical and always containing the c.g. of the ship; the x, z-plane, like the X, Z-plane, is assumed to be always in the undisturbed free surface, and the x-axis is taken along the tangent to the ship's course (defined as the projection on the X, Z-plane of the moving c.g. of the ship).

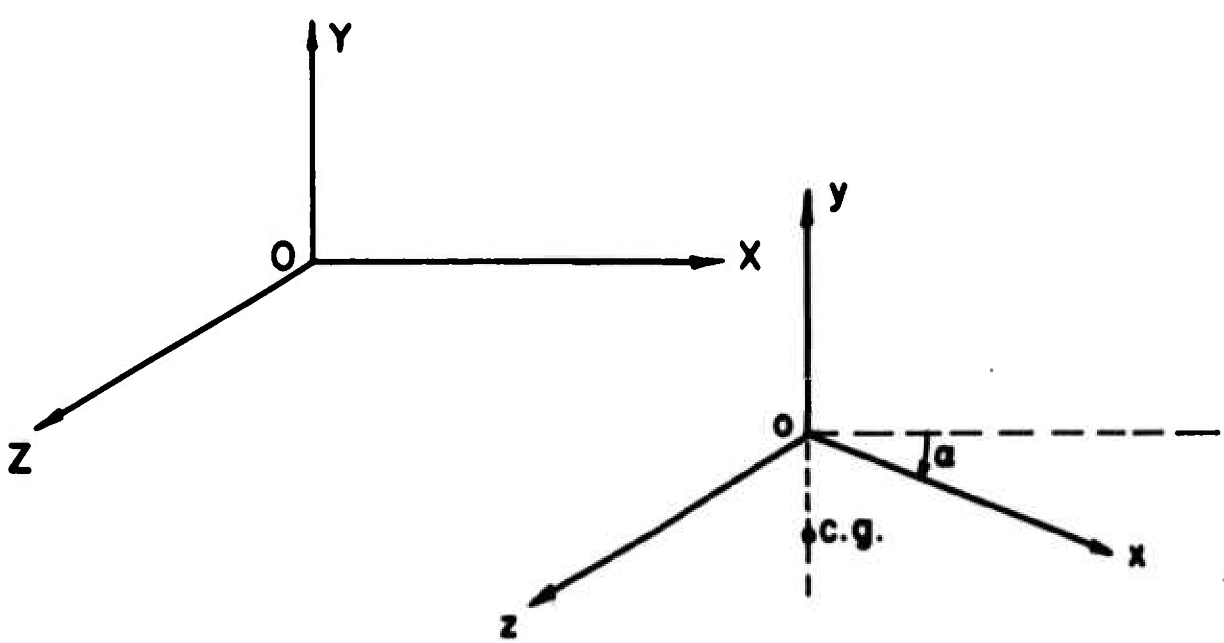
The forward speed $s(t)$ of the ship is by definition the magnitude of the velocity vector of the origin 0 of the moving coordinate system. The angular speed $\omega(t)$ of the moving system is

[3]Once the possibility of various types of developments is seen, some investigators are tempted to the conclusion that it might be a good idea to combine the advantages of various types of approximations by making developments with respect to two or more independent parameters, while here a development in terms of one parameter only is made. From the mathematical point of view such procedures are dubious, not to say dangerous, particularly if the dependence on the parameters is asymptotic and not analytic. For example, consider what might be called a clear-cut example, i.e. the Bessel functions $J_1/\epsilon_1 (\frac{x}{\epsilon_2})$, and the behavior with respect to ϵ_1 and ϵ_2 is to be deciphered for ϵ_1 and ϵ_2 close to zero. As is well-known, the behavior of these functions with respect to the two parameters is asymptotic in character and it is quite complicated. In fact it depends essentially upon how ϵ_1 and ϵ_2 are related as they tend to zero - thus one is really back again to the case of dependence on one parameter. On the other hand, as an applied mathematician, I would not rule out any procedure that is proved to work when tested by actual experience - but that is something different from asking whether a given procedure is valid from a rigorous mathematical point of view.



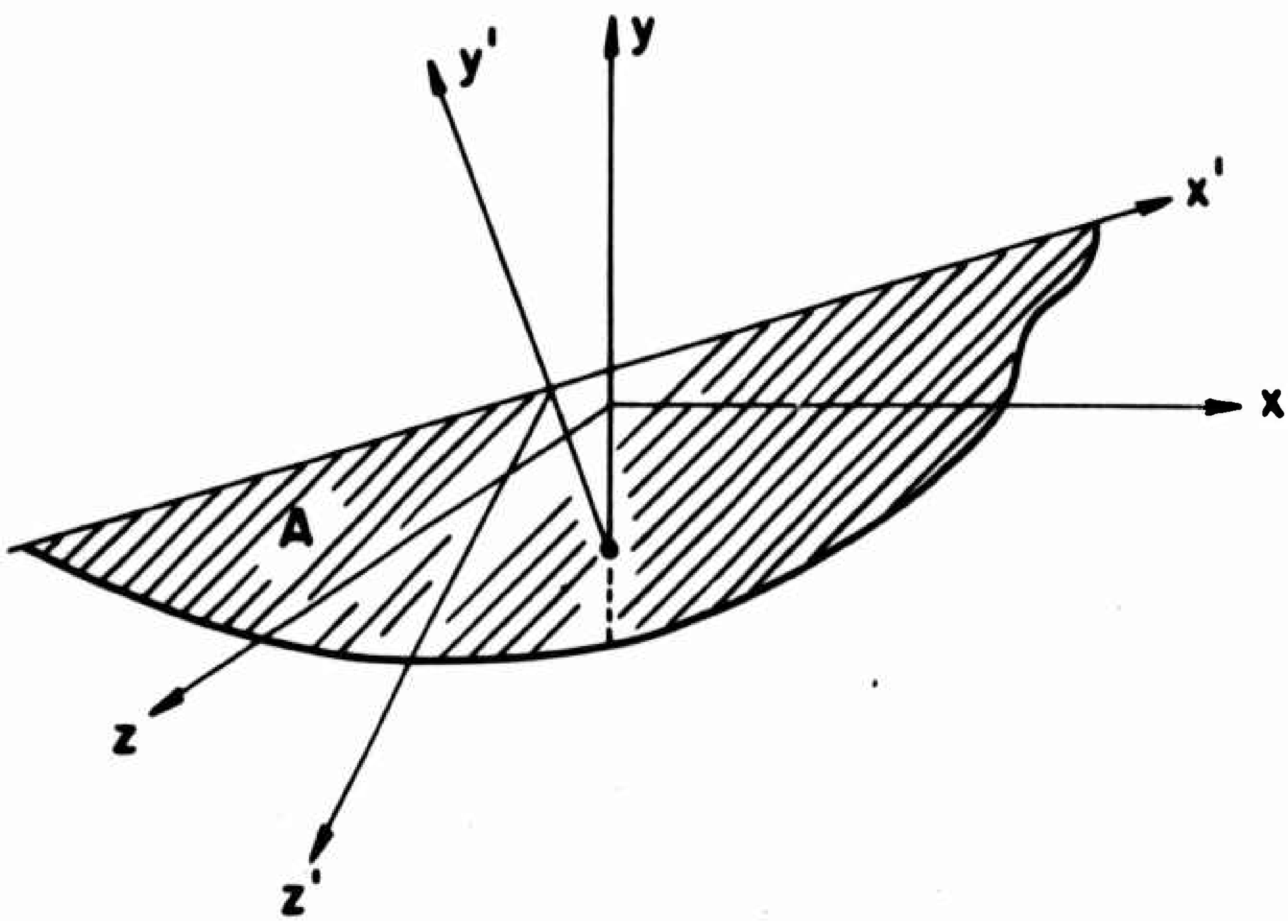
TYPES OF HULLS

Figure 1



FIXED AND MOVING COORDINATE SYSTEMS

Figure 2



THE TWO MOVING COORDINATE SYSTEMS

Figure 3

fixed in terms of the angle $\alpha(t)$ shown in Figure 2 by the relation

$$\omega(t) = \frac{d\alpha(t)}{dt}, \qquad \text{or} \qquad \alpha(t) = \int_0^t \omega(\tau)d\tau \qquad (2.1)$$

is α is defined to be zero for $t = 0$ (which implies that the systems X, Y, Z and x, y, z differ only by a translation at the initial instant).

Figure 3 indicates another moving coordinate system x', y', z' attached to the ship. The x',y'-plane coincides with the vertical lolongitudinal mid-section of the ship's hull (assumed to be symmetrical with respect to the plane) and is assumed to contain the c.g. The x'-axis is taken at the water line in the rest position of equilibrium of the ship, and the two systems x, y, z and x', y', z' are assumed to coincide in this position. The c.g. of the ship thus is not in general at the origin of the coordinate system x', y', z' fixed in it; the quantity y'_c is by definition the vertical coordinate of the c.g. in the primed system.

As was stated above, the physical assumptions on the flow are such that a velocity potential $\Phi(X,Y,Z;t)$ exists. It is a harmonic function of the space variables that is determined uniquely by imposing appropriate boundary conditions on the free surface, over the ship's hull, and at infinity; in addition, if unsteady motions are in question, it is necessary to impose initial conditions at the time $t = 0$. As was remarked in the introduction, this presents an extremely difficult non-linear initial-boundary value problem that is beyond the power of modern analysis to solve (including, probably, numerical analysis plus the employment of the best modern calculating machines).

The method of attack to be discussed in this talk is an approximate method based on a development with respect to a small slenderness parameter. For the generalization of Michell's theory the slenderness parameter is introduced by the relation

$$z' = \pm \beta h(x',y'), \qquad z' \gtrless 0, \qquad (2.2)$$

which gives the equation of the hull of the ship in the primed system of coordinates; the function $h(x',y')$ is defined over the projection A of the ship's hull on the x',y'-plane. Thus β is a dimensionless parameter which is the ratio of two lengths: one a length characterizing the beam of the ship, other other a significant length in the

direction of the keel, say, and hence it could be regarded as a beam-length ratio. The whole idea of the approximate method is to regard the velocity potential Φ and the quantities that fix the motion of the ship as functions of the parameter β and to develop then all in powers of β; the successive coefficients of the powers of β, it turns out, then are determinable as solutions of linear problems, and, what is also a highly important simplifying feature, in a fixed domain of the space variables even though the existence of a free surface deprives the original nonlinear problem of this characteristic. Thus we assume the following developments. For the velocity potential Φ we write

$$\Phi(x,y,z;t;\beta) = \beta\phi_1(x,y,z;t) + \beta^2\phi_2(x,y,z;t) + \ldots \quad . \tag{2.3}$$

The free surface elevation $\eta(x,z;t;\beta)$ of the water, and the speed $s(t;\beta)$ and angular velocity $\omega(t;\beta)$ are assumed to prove the developments

$$\eta(x,z;t;\beta) = \beta\eta_1(x,z;t) + \beta^2\eta_2(x,z;t) + \ldots \quad , \tag{2.4}$$

$$s(t;\beta) = s_o(t) + \beta s_1(t) + \beta^2 s_2(t) + \ldots \quad , \tag{2.5}$$

$$\omega(t;\beta) = \omega_o(t) + \beta\omega_1(t) + \ldots \tag{2.6}$$

Finally, the vertical displacement $y_c(t;\beta)$ of the c.g. of the ship and that its small angular displacements Θ_1, Θ_2, Θ_3 are assumed to be given by

$$\Theta_i(t;\beta) = \beta\Theta_i(t) + \beta\,\Theta_{i2}(t) + \ldots \quad , \tag{2.7}$$

$$y_c(t;\phi) = y_c' + \beta y_1(t) + \beta^2 y_2(t) + \ldots \tag{2.8}$$

It is important to observe that the forward speed $s(t;\beta)$ of the ship is assumed to have a development beginning with a term of zero order, corresponding to the fact that the forward speed of the ship is finite. (The assumption of an infinitesimal forward speed would be very restrictive, since the ship resistance, as given by the Michell theory, would then be zero.) All other quantities are found to have developments in which the term of zero order is not present; this will be exemplified in the case of the angular velocity $\omega(t;\beta)$ in (2.6) where this term is included - it will, however, be seen later that $\omega_o = 0$ results from the formal calculations. Thus we may speak of the angular displacements Θ_1, Θ_2, Θ_3 as the components of an angular displacement vector since they are infinitesimal.

In the end, the theory is to be developed in terms of the x, y, z-coordinate system; this requires that the formulas for coordinate changes from the primed to the unprimed systems must also be developed in powers of β. For example, the important special case of the equation of the ship's hull takes the form:

$$z + \beta\Theta_{21}(t)x - \beta\Theta_{11}(t)(y - y_c') - \beta h(x,y) + \ldots = 0 , \qquad (2.9)$$

after terms of order β^2 and higher are rejected.

In describing the motion of the ship it is important to put the notation introduced here in relation to the usual terminology. The angular displacements are named as follows: Θ_1 is the rolling, $\Theta_2 + \alpha$ the yawing, Θ_3 the pitching oscillation. The quantity $\beta s_1(t)$ in (2.5) is called the surge, $y_c - y_c'$ (cf. (2.8)) fixes the heave, and

$$\delta z = s_o\alpha = \beta s_o \int_0^t \omega_1(\tau)\, d\tau \qquad \text{fixes the sway.}$$

In addition, there are two quantities that fix what is called the trim of the ship, i.e. the time-independent values of the heave and the pitching angle that would result for a ship moving with a constant forward speed over a calm sea. Thus the six oscillation components, the two quantities fixing the trim, and the speed s_o determined by the propeller thrust, are nine quantities that serve to determine the motion of the ship.

A brief summary of the results obtained by way of generalization of Michell's theory follows: The generalization is obtained by insertion of the assumed developments (2.2) to (2.8) in all of the equations of the basic nonlinear problem, including those which formulate the boundary conditions as well as the differential equations. For example, it is necessary to evaluate $\Phi_x(x,y,z;t;\beta)$ on the free surface $y = \eta(x,z;t)$ in order to write down the boundary condition there. The calculation is the following:

$$\Phi_x(x,\eta,z;t;\beta) = \beta[\Phi_1(x,0,z;t) + \eta\Phi_1(x,0,z;t) + \ldots]$$

$$+ \beta^2[\Phi_{2x} + \eta\Phi_{2y} + \ldots] + \beta^3[\ldots.]$$

$$= \beta\Phi_1(x,0,z;t) + \beta^2[\eta_1\Phi_{1xy}(x,0,z;t)$$

$$+ \Phi_{2x}(x,0,z;t) + \ldots] \ldots \quad . \qquad (2.10)$$

We note the important fact that the coefficients of all powers of β are of necessity evaluated at $y = 0$, i.e. at the undisturbed equilibrium position of the free surface, and this comes about because the development for η is inserted in $\Phi_x(x,\eta,z;t;\beta)$, after which Φ_x is developed in powers of β.

In the same way, it turns out that the consistent way to satisfy the boundary condition imposed on the hull of the ship - and to all orders in β - is to satisfy it on the projection A of the hull on the vertical plane at each stage of the approximation. In our paper Peters and I pointed out that this observation settles, from a strictly mathematical point of view, an old controversy in which some investigators criticize the Michell theory because it does not satisfy the boundary condition on the curved surface of the hull. In this symposium the same argument has come up, so that it is perhaps worthwhile to state again our view on this matter. It is clearly not consistent to satisfy the boundary condition at the actual position of the hull while linearizing the problem otherwise since the forward velocity is finite and a hull of finite beam must then create waves of finite amplitude, thus making the linearization of the free surface condition (including satisfying it on the horizontal plane) of dubious validity. In effect, such a procedure amounts to taking into account some, but only a small fraction, of the relevant higher order terms in β (of which there are a huge number). However, it could be that such a procedure might, with luck, furnish the most essential contributions of higher order and thus indeed yield trustworthy and more accurate results. There is even an argument that lends this a certain plausibility, i.e. that the linear theory for surface waves seems to mirror some facts (propagation speed, and shape of the waves, for example) with good accuracy even if the wave amplitude is rather large. Once more we have here the question of the value of a given theoretical procedure from a practical point of view, which should then be tested

by experiment; certainly, mathematical arguments justifying it in the manner under discussion here would be hard to come by.

The results of the development with respect to β, up to terms of the order needed to obtain significant results are as follows - for the case of the generalized theory. The differential equation for Φ_1 is of course the Laplace equation:

$$\nabla^2\Phi_1 = 0 \ , \tag{2.11}$$

valid for y = 0 and outside of the vertical plane disk A which is the orthogonal projection of the hull on the x,y-plane in the equilibrium position. The boundary conditions resulting from the hull are:

$$\begin{aligned} \Phi_{1z} &= -s_o(h_x - \theta_{21}) - (\omega_1 + \dot{\theta}_{21})x + \dot{\theta}_{11}(y - y_c') \quad \text{on } A_+, \\ \Phi_{1z} &= \ \ s_o(h_x + \theta_{21}) - (\omega_1 + \dot{\theta}_{21})x + \dot{\theta}_{11}(y - y_c') \quad \text{on } A_-, \end{aligned} \tag{2.12}$$

with $A_{\pm}$ referring to the sides $z = 0_{\pm}$ of A; they express the fact that the flow is tangential to the hull. It is important to note that $s_o(t)$, $\theta_{21}(t)$, $\omega_1(t)$, $\theta_{11}(t)$ are functions that describe in part the motion of the ship. The free surface conditions are

$$\begin{aligned} -g\eta_1 + s_o\Phi_{1x} - \Phi_{1t} &= 0 \ , \\ -\Phi_{1y} - s_o\eta_{1x} + \eta_{1t} &= 0 \ , \end{aligned} \tag{2.13}$$

to be satisfied at y = 0. The first of these arises from the condition that the pressure on the free surface is zero, the second is the kinematic free surface condition.

Since the motion of the ship is to be determined by the forces acting on it, including the pressure of the water on its hull, it turns out to be convenient to work with the dynamic pressure $P(x,y,z;t;\beta) + gy$ and to develop it also in powers of β; to the lowest order it is given in terms of Φ_1 by

$$P_1(x,y,z;t) = s_o\Phi_{1x} - \Phi_{1t} \ , \tag{2.14}$$

and hence it, like Φ_1, is a harmonic function. The boundary conditions (2.12) and (2.13) - after elimination of η_1 from the second -, yield the following boundary conditions for P_1:

$$P_{1z} = \mp s_o^2 h_{xx} - s_o \omega_1 - 2s_o \dot{\theta}_{21} + x(\dot{\omega}_1 + \ddot{\theta}_{21}) - (y - y_c')\ddot{\theta}_{11} \quad \text{on } A_{\pm} , \tag{2.15}$$

and

$$s_o^2 P_{1xx} - 2s_o P_{1xt} + P_{1tt} + gP_{1y} = 0 \qquad \text{on } y = 0 \tag{2.16}$$

A complete formulation of the problem requires the addition of the dynamical equations for the motion of the ship, and these must also be developed in powers of β. To this end it is necessary to carry the development to terms of second order in β, for the following reasons: the speed s_o of the ship is of zero order, the amplitude of the oscillations of the ship is of first order, and the ship resistance is of second order. Only the third of these may need comment. Since the mass of the ship is of order β - because its volume is of that order and its density is at least bounded -, and its accelerations are also of that order it follows that the resistance is of second order. Or, as one also knows, the wave resistance can be calculated in terms of the amplitude of the surface waves created by its motion, in which case it is well-known that it depends on the square of that amplitude. The fact that the development must be carried up to <u>second</u> order is one of the reasons why a careful (even though tedious) formal development is made rather necessary.

The terms of first order for the motion of the ship furnish the following relations:

$$\dot{s}_o = 0 , \tag{2.17}$$

$$2\rho g \int_A \beta h dxdy = M_1 \beta g , \tag{2.18}$$

$$\int_A \beta h x dxdy = 0 , \tag{2.19}$$

$$\int_A [P_1]_-^+ dxdy = 0 , \tag{2.20}$$

$$\int_A [xP_1]_-^+ dxdy = 0 , \tag{2.21}$$

$$\int_A [yP_1]_-^+ \, dxdy = 0 . \tag{2.22}$$

Here $M_1\beta$ is the mass of the ship, P_1 yields the dynamic pressure to first order, and the symbol $[\ ]_-^+$ means that the jump in passing from A_+ to A_- is to be taken.

The equation (2.17) says that the term of zero order in the speed is constant, and hence the motion in the x-direction is a small oscillation (the surge) relative to this constant speed. Equations (2.18) and (2.19) are expressions of static laws of equilibrium: the first is the law of Archimedes, and the second requires the center of buoyancy to be on the same vertical line as the c.g. The remaining three equations are dynamical equations which, through P_1 and the boundary conditions (2.15) and (2.16), contain three components of the oscillation of the ship, i.e. θ_{11}, θ_{21} and ω_1. This is quite interesting because it says that P_1 and these three oscillations are in principle determined without reference to any other components of the oscillation. In other words, the motion of the ship, which is fixed by P_1 is entirely independent of the pitching, heaving, and surging oscillations θ_{31}, y_1, and s_1. This confirms a point raised earlier, i.e. that these components of the oscillation create waves with amplitudes that are of second order at least, and hence they have no effect in determining P_1, which fixes a term of first order. In addition, it is to be expected that the oscillations determined by θ_{11}, θ_{21}, ω_1 will be damped out - i.e. damping in rolling, yawing, and swaying will occur - since these oscillations create waves having amplitude of first order which then can carry off energy to infinity.

Finally, the terms of second order in Φ lead to the following equations:

$$M_1\dot{s}_1 = \rho \int_A h_x[P_1^+ + P_1^-] \, dxdy + T, \tag{2.23}$$

$$M_1\ddot{y}_1 = -2\rho g \int_L (y_1 + x\theta_{31})h dx + \rho \int_A h_y[P_1^+ + P_1^-] \, dxdy, \tag{2.24}$$

$$I_{31}\ddot{\Theta}_{31} = -2\rho g\Theta_{31}\int_A (y - y_c')h\,dx\,dy - 2\rho g y_1 \int_L xh\,dx$$

$$- 2\rho g\Theta_{31}\int_L x^2 h\,dx + \ell T + \rho\int_A [xh_y - (y - y_c')h\,][P_1^+ + P_1^-]dx\,dy\,. \tag{2.25}$$

We note that line integrals of the form $_L\int \ldots dx$ occur; they refer to integrals over the projected water line L of the hull. The equation (2.23) serves to determine the surge $s_1(t)$ and, by taking the time independent terms, the forward speed s_0 in terms of the propeller thrust T. (One might also regard s_0 as given at the outset, in which case T, the wave resistance of the ship, would be determined.) The remaining pair of equations serves for the determination of the pitching and heaving oscillations (and also the trim through the time-independent terms). One observes that the various components of the oscillation of the ship are coupled in rather complicated fashion with each other and with the motion of the water.

Equations (2.23), (2.24), and (2.25) confirm an important conclusion drawn earlier, i.e. that no damping of the surging, heaving, and pitching oscillations occurs in consequence of this theory since these second order linear ordinary differential equations have no terms in the velocities (i.e. the first derivatives of the surging, heaving, and pitching displacements). That this should come about inevitably was argued earlier on physical grounds.

It has already been said that the theory just derived is a generalization of Michell's theory, which had as its main purpose the derivation of an integral formula for the wave resistance. In fact, it is possible to solve a more general problem explicitly, i.e. that in which the ship oscillates only in the vertical plane and is travelling through a train of waves with crests at right angles to the course of the ship. Upon examining the boundary conditions (2.15) and (2.16) one sees that they simplify remarkably for this type of motion, since all of the terms referring to the motion of the ship are zero except s_0. But this means that the boundary problem for P_1 (and hence for the velocity potential Φ_1) is exactly the same as it was for Michell's problem, and one finds (cf. the paper by Peters and me, p. 445) that equation (2.23) then yields Michell's formula for the wave resistance. The equations (2.24) and (2.25) thus can also be solved explicitly, but the integrals

involved are much more complicated than Michell's integral. It is thus clear that the theory presented here is indeed a generalization of Michell's theory.

Finally, this special case gives rise to another pertinent observation, i.e. that resonance is possible at certain frequencies of the wave trains in which the ship is moving. The simplest case arises if one assumes that $\int_L xhdx = 0$, i.e. that the c.g. of the area of the horizontal section of the hull at the water line is on the same vertical as the c.g. of the whole ship. In this case the differential equations (2.24) and (2.25) simplify remarkably, as follows:

$$\ddot{y}_1 + \lambda_1^2 y_1 = p(t) , \tag{2.26}$$

$$\ddot{\theta}_{31} + \lambda_2 \theta_{31} = q(t) , \tag{2.27}$$

with

$$\lambda_1^2 = \frac{2\rho g \int_L hdx}{M_1} , \tag{2.28}$$

$$\lambda_2^2 = \frac{2\rho g \left[\int_A (y-y_c')hdxdy + \int_L x^2 hdx \right]}{I_{31}} . \tag{2.29}$$

As one sees, these oscillations are not coupled, and two resonance frequencies given in terms of λ_1 and λ_2 occur which are fixed by the geometry of the hull. The fact that no damping occurs is also clearly evident.

3. Ships of Small Draught-Length Ratio

It was already indicated that hulls of this type - referred to earlier and in Figure 1 as of the planing type - lead to a theory with markedly different characteristics than that given by the generalized Michell theory, above all with respect to which modes are damped. In view of earlier discussions it seems reasonable to state the facts with little comment.

For ships of this type the equation of the hull with respect to the primed system of coordinates is best given by

$$y' = -\beta h^*(x',z') \tag{3.1}$$

with the function h* defined over the cross-section B of the hull in the equilibrium position that is cut out by the horizontal plane $y = 0$. Thus β may be considered to be a representative draught-length ratio. The solution of the problem is then developed in powers of this parameter, with the following results.

The dynamic pressure P_1 to first order satisfies the boundary condition:

$$P_{1y} = s_o(s_o h^*_{xx} + 2\dot{\theta}_{31}) - \ddot{y}_1 + \dot{\theta}_{11}z - \dot{\theta}_{31}x \quad \text{on } B . \qquad (3.2)$$

The free surface condition is the same as (2.16), though it now applies only to the part of the plane $y = 0$ outside the domain B. Thus the determination of P_1 depends upon either prescribing or otherwise determining the four quantities $s_o(t)$, $\theta_{31}(t)$, $y_1(t)$, and $\theta_{11}(t)$; in particular, it is now the pitching, heaving, and rolling oscillations that are in question. In accordance with the introductory discussion, and by analogy with what was found in the previous case, it is to be expected that only these oscillations will appear to first order when the quantities specifying the motion of the ship are developed with respect to β; while the yawing, swaying, and surging oscillations are expected only from terms of second order. And so it turns out.

To first order the following equations for the motion of the ship are found:

$$-\rho g \int_B (y_1 - \theta_{11}z + \theta_{31}x)dxdz + \rho \int_B P_1 dxdz = 0 , \qquad (3.3)$$

$$\rho g \int_B (y_1 - \theta_{11}z + \theta_{31}x - h^*_x)zdxdz - \rho \int_B P_1 zdxdz = 0 , \qquad (3.4)$$

$$-\rho g \int_B (y_1 - \theta_{11}z + \theta_{31}x - h^*_x)xdxdz + \rho \int_B P_1 xdxdz = 0 . \qquad (3.5)$$

As one sees, it is y_1, θ_{11}, and θ_{31} that occur, and hence these equations, together with the equations for P_1 furnish the proper number of conditions to fix all four of them.

The equations of motion for the remaining components of the oscillation of the ship result from consideration of terms of order β^2. They lead to second order differential equations analogous to (2.23), (2.24), (2.25) for the quantities $s_1(t)$, $\theta_{21}(t)$, and $\omega_1(t)$ - that is, for the surging, yawing, and swaying oscillations. (The ship resistance in relation to the forward speed s_0 also is obtained by consideration of the time-independent terms in the equation of motion for the x-direction.) These oscillations are not damped, but those of the previous group, i.e. the heaving, pitching, and rolling oscillations, are damped.

Thus, this theory, as was already remarked, may well be more in accord with observations than the Michell theory since it in principle yields damping in pitching and heaving, and also for most ships the draught-length ratio is smaller than the beam-length ratio. Unfortunately, no explicit solutions of any kind have been obtained using this theory, and it would seem not too likely that any simple ones exist. Not even the analogue of the wave resistance integral of Michell is found.

In our paper, Peters and I formulated this theory and the Michell theory in terms of integral equations with the aid of an appropriate Green's function. These are integral equations over the plane disk A in the Michell-type problem and over the disk B in the present case. This is, at least on a superficial view, rather reasonable since then a linear problem over a finite and two-dimensional domain is substituted for a three-dimensional problem in an infinite domain in which radiation conditions would have to be satisfied at ∞. However, the integral equations are singular, the Green's function is given by a complicated two-fold integral (and thus the integral equations themselves are in general four-fold integrals), and there are singularities along the sharp edges of the disks. It seemed to us necessary, therefore, to make sure, if possible, that the integral equations, which would quite clearly be satisfied by a solution of the original problem, have uniquely determined solutions. This was done in our paper, once rather reasonable conditions from the physical point of view were imposed at the edges of the disks A and B. It turned out that the problems of Michell's type are more complicated than those for planing ships since the Green's function is simpler in the latter case, and also the uniqueness theorem for planing ships could be derived for all cases while for the Michell-type ships it was proved only in case the ship was oscillating without forward speed, or had a forward speed but was not oscillating. Thus, the planing ship problem may be the one of the two more likely to yield to attack

by numerical methods, for example, though it seems to have no explicit solutions. We therefore think it would be of interest to produce some numerical solutions for planing ships - it might well be that the ship resistance, for example, would be given more accurately by such a procedure than it is by Michell's formula.

4. Yacht-Type Hulls, Slender Body Theory, Kelvin's Theory

Finally, it seems worthwhile to discuss three further theories, though briefly.

The yacht-type hull of Figure 1 leads to a theory developed by Peters and me which combines the features of the two previous cases. Now all modes except the surging mode of oscillation are damped - in accord with the fact that all of these modes of oscillation have the effect of pressing a flat face against the water. The integral equation for these cases is of course defined over both disks A and B, and it is quite complicated.

In this symposium there has been a good deal of discussion of what is called slender body theory. This means that the ship's hull is regarded as slender with respect to both beam and draught, and hence the hull is regarded as being collapsed into a line segment. Thus the effect of the hull must be replaced by an appropriately chosen distribution of singularities along the line segment. Such an approximation is of course much more drastic than those proposed by Michell and by Peters and me, and consequently a great deal is lost. For example, no damping in any mode occurs. On the other hand, it might be that the ship resistance could still be obtained with reasonable accuracy in this way.

Kelvin's theory of ship waves is not usually mentioned when discussing actual ships, but it is neverthelsee pertinent. In this theory, the most drastic approximation of all is made, i.e. the entire hull of the ship is assumed to be collapsed into a single point, and the ship is thus regarded as a moving point singularity. There is thus only one parameter left to describe the ship, i.e. the strength of the singularity, and at first sight one might feel that such an approximation would furnish little or nothing of value. However, that is not so. Kelvin's theory yields, even when it is treated approximately by means of the method of stationary phase, important aspects of the wave pattern created by the ship that are

in very good accord with observation. For example, the existence of two different wave trains, the confinement of the disturbance within a sector with an angle fixed independent of the speed, the speed-wave length relation, the phase relation between the two different types of waves - all are correctly given by this theory. To my mind, the fact that such a drastic approximation gives these results leads one to hope that such theories as that of the planing ship might given good and useable results.

ENERGY RELATION FOR THE WAVE RESISTANCE OF SHIPS IN A SEAWAY

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ENERGY RELATION FOR THE WAVE RESISTANCE OF SHIPS IN A SEAWAY

INTRODUCTION

The mechanism by which the resistance is increased under the influence of ocean waves has been discussed by various researchers for more than twenty years, but no theory evolved could give a consistent description to the source of the resistance increase until recent years. This was a result of the complicated nature of the fluid motion around a ship in a seaway which had been defying the mathematical analysis. The first attempt at a theoretical approach seems to be attributable to Havelock.[1] He discussed the horizontal force acting on a thin ship moving among waves with uniform velocity. His conclusion was that the extra force due to the existence of the wave was periodic, so there was no resistance augmentation in the order of approximation. An attempt was made by Kreitner[2] who proposed a semi-empirical formula which gave the increase of resistance in terms of the wave reflection at the surface of the ship. An analytical and rigorous calculation was made by Havelock[3] with respect to the average force which was brought by the reflection of waves. It showed a resistance too small to give the extra resistance of a ship among waves. Some years later, Havelock[4] made an attempt to give the resistance augmentation in a different way. The close correlation of increase of resistance with the ship's oscillations, especially pitching, has been recognized as a matter of fact, since Kent's[5] resistance experiments in regular waves. Havelock was the first one who established a theory which gave the relationship between the increase of resistance and the oscillation of ships, though the basic idea by which Havelock's theory was developed was due to Watanabe.[6] The latter proposed a theory of the drifting force experienced by a ship rolling in beam seas. Therefore, the theory may be called the drifting force theory. It assumes the force acting on the ship hull to be calculated from the fluid pressure in the ocean wave which is not disturbed by the ship. This assumption is similar to the hypothesis of Froude[7] and Krilov[8] in the theory of ship's oscillations. According to the drifting force theory, the ship experiences a continuous force independent of time in the direction to which the waves propagate, if a phase lag exists between the oscillation of the ship and the excitation by the wave, even though the pressure itself is purely periodical. Havelock applied it to the heaving and pitching of a ship

in head seas and found a simple relation between the resistance augmentation and the amplitude and phase of the oscillations. At one time, Havelock's theory was regarded as the most reliable one of existing theories about the resistance increase due to the seaway and some numerical examples were reported by St. Denis.[9] However, physical significance was still obscure because of the formalism in the deduction of the formula, and there was opinion that the underlying principle of the theory was quite doubtful, such as the criticism by Russian scientists[10] who blamed it as violation of the principle of energy conservation. There is another theory derived in a different way. Hanaoka[11] studied the resistance when a ship was moving in a calm sea with a uniform velocity but making heaving and pitching motions under the excitation of some external mechanism such as an oscillator. He called the increment of resistance in this case the nonuniform wave resistance. He showed that the principle of energy conservation was attained in this case and some years later, this was reproduced by Newman.[12] One may feel some confusion by the coexistence of two theories, which stand on quite different theoretical bases.

This short essay aims to unravel the entanglement between two theories and give a lucid answer to the origin of the resistance augmentation due to the seaway.

The Energy Consideration for a Ship in Regular Head Seas

Though the fluid motion near the ship is very complicated, the fluid motion far from the ship shows a comparatively simple nature, and an energy analysis is possible which gives correct information about the resistance experienced by the ship. This sort of analysis was employed first by Havelock[13] in discussing the wave resistance of ships in a calm sea. Havelock assumed two vertical planes, one of which in front of a ship moving on a calm sea with uniform velocity and the other far behind it, and derived the wave resistance from the energy variation in the fluid contained between these planes. Recently Newman extended Havelock's method of analysis to an oscillating ship. A similar analysis can be applied to the ship in a seaway, but as the waves generated by the ship also propagates sideways in this case, vertical planes are not suitable as an energy control surface. Instead of vertical planes, a cylindrical surface with a vertical axis and radius large enough to surround the ship is taken as the control surface.

In the first place, the cylindrical surface is assumed to be fixed and the ship is moving with a uniform velocity U. After a unit time has elapsed, the ship comes to the position of distance U from the initial position. Now we compare the energy of the water inside the cylinder at the beginning and that at the latter instant. If we assume for the moment that the fluid motion around the ship is stationary in order to simplify the exposition, the change of the energy inside the cylinder is attributed solely to the change of the position of the ship.

If, on the other hand, the cylinder is assumed to move with the ship, the cylinder will come to a new position at a distance U from the original one after a unit time. We will designate the initial position of the cylinder by C_1 and the new position by C_2. As the fluid motion around the ship is assumed stationary, the energy inside the surface C_2 at the latter instant must be the same as that inside C_1 at the beginning.

Now, returning to the former case that the cylinder is fixed at the position C_1, the energy inside it changes as the ship moves to the new position. Here let us designate the space inside C_1 excluding the portion inside C_2 by I, the space inside C_2 excluding the portion inside C_1 by II, and the common space inside both C_1 and C_2 by III. The energy contained in each portion after a unit time has elapsed is designated by E_I, E_{II} and E_{III} respectively. As the fluid motion relative to the coordinates fixed to the ship is assumed unchanged, the energy inside C_1 at the beginning and that inside C_2 after a unit time must be the same. The latter is $E_{II} + E_{III}$. Since the energy inside C_1 after a unit time is $E_I + E_{III}$, the net variation of the energy in a fixed cylinder at C_1 during a unit time is given by

$$\partial E/\partial t = E_I + E_{III} - (E_{II} + E_{III}) = E_I - E_{II} \qquad (1)$$

If the cylinder is assumed to move with the ship, the above quantity is identical with the energy efflux which is carried away by the fluid going across the surface. As the energy inside the fixed cylinder changes, the net variation of the energy must be supplied from outside according to the principle of energy conservation. The energy supply is partly brought in by the fluid flow at the boundary, i.e., the cylindrical surface. We shall write this $\partial\bar{W}/\partial t$. It is positive if the energy flows inwards across the surface, but it may become negative in some cases. Then the difference $\partial E/\partial t - \partial\bar{W}/\partial t$ must be supplied by the ship. When a ship is towed by a constant force which overcomes the resistance R, the effective work done in a unit

time is RU and is the rate of the energy supplied to the fluid. Then the principle of the energy conservation gives the following relation.

$$\frac{\partial E}{\partial t} = \frac{\partial \overline{W}}{\partial t} + RU \tag{2}$$

This relation holds also in the case where viscous forces exist, if the energy dissipation by the viscosity is taken into account. In the present analysis, however, the effect of viscosity is assumed unimportant so that the energy is carried away only by the potential motion of the fluid. We can write

$$R = \frac{1}{U} \left(\frac{\partial E}{\partial t} - \frac{\partial \overline{W}}{\partial t}\right) \tag{3}$$

where $\partial E/\partial t$ and $\partial \overline{W}/\partial t$ are both determined by the fluid motion at the cylindrical surface if the motion is stationary. If we take the radius of the cylinder very large, the resistance is determined by the fluid motion at a great distance from the ship.

Now turning to the ship among waves, the motion of the ship and that of the fluid changes from time to time. Even in the non-stationary motion, the relationship between the resistance and the energy variation given before still holds in principle, but the term $\partial E/\partial t$ cannot be determined by the fluid motion at the cylindrical surface alone because the fluid motion within the cylinder moving with the ship changes periodically. As the resistance of a ship among waves is understood in the sense of time-average of the horizontal force, the time-average of each term of the equation is sufficient to the analysis of the resistance. If the motion is periodic the time-average has a definite value which is independent of time, so that the energy relation is discussed similarly to the case of calm sea.

Let us consider a ship moving under a constant towing force in regular head seas. Though the forward speed changes periodically, the speed of advance is defined as the average velocity of the ship. The periodical change of the velocity is a component of the ship's oscillations, i.e., surging. In general, the ship makes oscillations with six degrees of freedom by the action of the wave, but in head seas, the oscillations have three modes, heaving, pitching and surging. Instead of a ship moving with forward velocity U, we first consider

the case that a ship is floating on a stream of velocity U in the opposite direction to the advance velocity of the ship, the average position of which is sustained at a fixed position by a constant towing force, and a regular train of waves is superimposed on the uniform stream. As the towing force keeps the ship at a stationary position, it does not work on the average. Therefore there is no external force that supplies energy, except the hydrodynamic action.

Let us consider two modes of oscillations, heaving and pitching, the amplitudes (double) of which are Z and Ψ respectively. The heaving is excited by the periodic force of amplitude F and the pitching by the periodic moment of amplitude M. There are phase lags between the motion and the excitation. They are ϵ_1 for heave and ϵ_2 for pitch. The phase lag in the oscillation is the consequence of the damping term in the equation of motion. The damping force and moment consume the energy. In the absence of the viscous forces, the energy is transmitted by the wave which is possibly generated by the ship. The energy consumed or transmitted by the wave in a complete cycle can be calculated at once as follows:

$$(\pi/4)\ FZ \sin \epsilon_1 + (\pi/4)\ M\Psi \sin \epsilon_2$$

So the average rate of work which must be done to maintain the motion is

$$P_1 = \frac{\pi}{4T_e} FZ \sin \epsilon_1 + \frac{\pi}{4T_e} M\Psi \sin \epsilon_2 \qquad (4)$$

where T_e is the period of encounter. Since the motion of the ship is excited by the wave pressure and no other source of energy exists, the energy must be supplied by the incident waves. If the incident wave is not disturbed by the ship, or there is no scattered wave, the energy brought into the cylinder by the wave is carried away by the same wave out of the cylinder. Therefore there is not supply of energy. Consequently the scattered wave again plays the role of supplying energy. If the wave is produced by an oscillating ship in calm water, the wave can only disperse the energy. Accordingly, the scattered wave cannot supply energy by itself. The only possibility of energy is the interference between the incident wave and the scattered wave. The principle of the energy conservation indicates that P_1 is identical with the energy taken into the cylinder per unit time as a result of the interference between the incident wave and the scattered wave.

Next, let us consider another case in which there is a uniform stream of the same velocity as the wave celerity c but the direction of the flow is opposite to the propagation of the regular wave. In this case the wave superimposed on the uniform stream does not propagate and steady motion of the fluid is obtained. The motion of the ship, which corresponds to uniform motion with velocity U in the opposite direction of the propagation of wave of celerity c, is the motion with uniform velocity $U + c$ in the system now being considered. Here the regular train of incident wave does not propagate but form just a wavy surface of the stream upon which the ship is moving. Accordingly, there is no energy transmitted by the incident wave. The energy is carried away across the vertical cylinder which encircles the ship by the wave generated by the ship and there is nothing to compensate for it but the work done by the towing force T which overcomes the resistance to propel the ship with velocity $U + c$. The wave generated by the ship is constituted of a wave system which accompanies the ship maintaining a constant pattern, and another wave system which is the consequence of the oscillatory motion and is itself periodic. As the energy transmitted by the wave is a quadratic functional of the wave motion, we can regard the resistance as a sum of two components corresponding to the energy transmission by each wave system.

Accordingly, we can write

$$T = R + \Delta R \tag{5}$$

The component ΔR is the extra resistance as a consequence of the seaway. The work done by the part of the towing force which overcomes ΔR is consumed by the transmission of energy through the periodic wave and its rate per unit time is identical with P_1. If there is no supply of energy elsewhere, the following relation exists.

$$\Delta R(U + c) = P_1 \tag{6}$$

There is a relation between the period of encounter and the wave length λ as follows in the case of head seas.

$$T_e = \frac{\lambda}{U + c} \tag{7}$$

If the waves generated by the ship are identical with those which accompany the heaving and pitching of the ship in calm sea, we can equate the right-hand side of Equation (4) with the left-hand side of Equation (6). Dividing both sides by $U + c$ and applying the

relation (7), we find the formula

$$\Delta R = (\pi/4\lambda)\ FZ \sin \epsilon_1 + (\pi/4\lambda)\ M\Psi \sin \epsilon_2 \qquad (8)$$

This equation coincides exactly with the formula given by Havelock, who derived it from the drifting force theory. Thus there is no contradition in the principle of energy conservation in spite of the criticism stated before. This relation is not, however, an exact one. The energy efflux P_1 corresponds to $- \partial\overline{W}/\partial t$ of Equation (3) in the present system of motion, but as the result of the ship's motion with velocity U + c, another term which corresponds to $\partial E/\partial t$ must be considered. Therefore we must write instead of Equation (6)

$$\begin{aligned} \Delta R\ (U + c) &= P_1 + \partial E/\partial t \\ &= P_1 + P_2 \end{aligned} \qquad (9)$$

We have assumed that the waves generated by the ship are identical with those produced by the heaving and pitching oscillations of the ship in a calm sea. If the oscillation of the ship is excited by some external mechanism such as an oscillator, the latter exerts work to overcome the damping forces and the average rate of work done in a unit time is P_1. Hanaoka studied this case and found a change in resistance against the forward motion as a result of the oscillation. Let us designate it by ΔR_2. The energy transmitted by the wave, which is generated by the ship is $P_1 + P_2$ in a unit time, while the energy supplied to the ship by the external forces, that consist of the towing force and the oscillator, is

$$P_1 + \Delta R_2\ (U + c)$$

Therefore, the following relation exists.

$$P_1 + \Delta R_2\ (U + c) = P_1 + P_2 \qquad (10)$$

That is

$$P_2 = \Delta R_2\ (U + c) \qquad (11)$$

Combining Equations (9) and (11), we obtain

$$\Delta R = \Delta R_1 + \Delta R_2 \qquad (12)$$

Thus the increase in resistance is the sum of the force obtained by the drifting force theory and the resistance due to the oscillation in a calm sea, the nonuniform resistance by Hanaoka.

It has been assumed in the above discussion that the waves generated by the ship are identical with those produced by the same ship constrained to oscillate in a calm sea. However, the waves accompanying a ship in a seaway are somewhat different from the above because of the undulation of the sea surface. We can take account of the effect of the seaway by assuming a fictitious deformation or snake-like motion of the ship hull. In fact, Hanaoka applied this technique when he attempted to extend his nonuniform wave resistance theory to the resistance among waves. Though this idea can be applied to the component ΔR_2, there is a difficulty in its application to ΔR_1, the drifting force.

Giving up the theory of drifting force and that of nonuniform wave resistance, we can obtain the total increment of the resistance ΔR directly from an energy analysis of the wave generated by the ship together with the incident wave. As a detailed analysis is given in Reference 14, there is no intention to reproduce the procedure of analysis here.

In order to obtain an analytical expression, the problem is analyzed by means of the linearizing procedure, assuming that the wave height is small compared with other quantities such as the wave length or the ship length. As the energy is a quadratic functional of the fluid velocity, the added resistance is of second order with respect to the wave height. Let us designate the ship's length by L, the beam by B, the wave amplitude (= 1/2 wave height) by r, the density of water by ρ and the acceleration of gravity by g. Then the following expression is proposed for the increment of the resistance.

$$\Delta R = \rho g r^2 \frac{B^2}{L} K_w \tag{13}$$

where K_w is a dimensionless coefficient which is a function of the wave length, the ship's speed and the ship form.

Resistance in an Irregular Seaway

The discussion in the preceding section has dealt with regular head seas, but a similar analysis applies to regular oblique seas and also to following seas. In order to apply the theory to the

actual seaway which is irregular and complicated, we assume that possibility of linear superposition. The seaway is assumed to be constituted of a superposition of plane regular waves of infinitesimal amplitudes with different frequencies arranged in various directions. The amplitudes of the component waves are distributed over any value of the frequency and direction, and the manner of the arrangement is defined by the energy spectrum of the sea. The assumption of linearity results in the ship oscillation being linear with respect to the motion of the sea surface. This means that the amplitude of any component of oscillation is obtained by a linear operation. In contrast the increase of resistance due to the seaway is a quadratic quantity with respect to the wave. If the amplitude of the wave element in the direction χ having the circular frequency ω is dr, the energy spectrum of the sea is defined in the following form

$$E(\omega, \chi)\ d\omega\ d\chi = (dr)^2 \tag{14}$$

The energy of the compound wave is represented by the integration of the spectrum

$$E^* = \int_0^\infty \int_0^\pi E(\omega, \chi) d\omega\ d\chi \tag{15}$$

where E* is called the cumulative energy density. The sea surface is expressed by

$$Z_\omega = \int_0^\infty \int_0^\infty \sqrt{E(\omega, \chi) d\omega\ d\chi} \quad \sin\left[\frac{\omega^2}{g}(x \cos\chi + y \sin\chi) + \omega t + \epsilon(\omega, \chi)\right] \tag{16}$$

where $\epsilon(\omega, \chi)$ is the phase of the wave element in the direction χ with the frequency ω. Since the ship's oscillations are linear quantities, the motion of the ship is a linear superposition of the oscillations excited by each wave element. The waves generated by the ship's oscillations are also linear and their superposition is possible. The frequency of the ship's oscillations is not ω but ω_e, the circular frequency of encounter. Accordingly, the frequency of waves generated by the ship must be defined by the frequency of encounter. Though ω and ω_e have a one-to-one correspondence in the plane head seas, there is no such relation in the three-dimensional irregular seaway. Though the energy spectrum of the seaway is defined in the ω-χ plane, the energy relation with which the resistance is concerned, must be discussed

in the ω_e-χ plane. In the transformation from ω_e-χ plane to ω-χ plane, ω is a multivalued function of ω_e. Then cross products between components of different ω but with the same ω_e give time-independent terms and do not vanish when the time-average is taken. This fact prevents an analytical expression for the time-average of the resistance in terms of the spectrum of the sea. If the problem is discussed in a statistical sense, the expression for the average increase of resistance can be obtained by assuming a random phase.[15]

We designate the increment of the resistance in a regular plane wave of amplitude dr, circular frequency ω and heading angle χ by the following form.

$$dR = \rho g (dr)^2 \frac{B^2}{L} K(\omega, \chi) \qquad (17)$$

Then the average increase of resistance in the irregular seaway defined by the energy spectru, Equation (14), is expressed as follows:

$$\Delta R = \rho g \frac{B^2}{L} \int_0^{\infty} \int_0^{2\pi} K(\omega, \chi)\, E(\omega, \chi)\, d\omega\, d\chi \qquad (18)$$

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THE APPLICATION OF FRIEDRICHS TECHNIQUE TO THE SHIP PROBLEM

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INTRODUCTION

The ship problem has recently received attention from many investigators, for example, Tuck,(1) Vossers,(2) Maruo(3) and Newman.(4) This paper describes the result of an investigation partially reported in Reference 5. We use a perturbation scheme for the solution of the ship problem which is very similar to the slender ship hydrodynamics of Tuck(1) and Vossers,(2) and obtain the same resistance formula found by them. The perturbation problem is then solved by making use of a Green's function derived by Peters and Stoker.(6,7) In addition we calculate the resistance of twin hull ships. We investigate the problem of optimum hulls and obtain for large Froude numbers a variational problem that is very similar to the problem solved by Von Karman for the shape of the projectile of fixed volume having minimum wave-resistance. The resulting variational problem has a solution, and we find, for example, that the optimum hull shape for an axisymmetric hull has a blunt radius distribution. We then calculate the wave resistance of these optimum hulls.

The Ship Problem

Consider a ship moving with velocity U through a semi-infinite ocean. Let the ship be held at a fixed attitude. Represent the ship in the coordinate system shown in Figure 1 by the relations

$$\begin{array}{ll} & -a^- \leq x \leq a^+ \\ y = y^\pm(x, z) & z^-(x) \leq z \leq z^+(x) \end{array} \qquad (1)$$

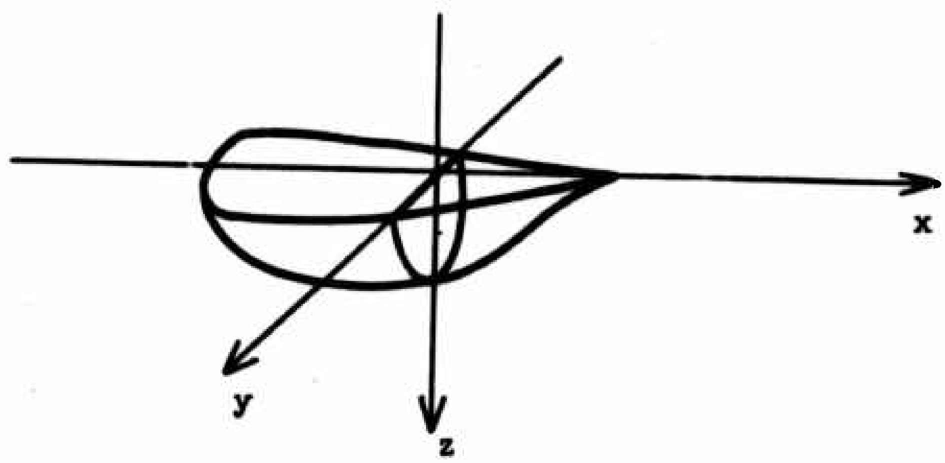


Figure 1

Let this ship move with velocity

$$(U,0,0)$$

through an incompressible fluid of density ρ in the semi-infinite region $z \geq 0$.

The equations of motion for the fluid are

$$\nabla \cdot \vec{q} = 0 \tag{2}$$

$$\rho \left[\frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla) \vec{q} \right] = - \nabla p + \rho g \vec{\epsilon}_z \tag{3}$$

with the boundary conditions

$$\vec{q} \cdot n = U \cdot n \qquad \text{on ship} \tag{4}$$

$$p = p_o \qquad \text{on free surface} \tag{5}$$

where $\vec{q}$ is the velocity vector, and n is the normal direction outwards from the fluid.

On the free surface the fluid velocity should be tangential to the surface because the fluid cannot move through this surface. Let

$$z = \eta(x, y, t) .$$

represent the free surface. The normal vector to this surface is in the direction of the vector.

$$(\eta_x, \eta_y, -1)$$

so that the condition that the fluid flow be tangential to the free surface is expressed by the relation

$$q \cdot (\eta_x, \eta_y, -1) = (0, 0, \eta_t) \cdot (\eta_x, \eta_y, -1) \quad .$$

The last relation may be written in the form

$$u\eta_x + v\eta_y - w + \eta_t = 0. \tag{6}$$

Equations (2) and (3) with the boundary conditions (4), (5) and (6) describe the hydrodynamics of the ship problem. We propose to study the solution of the ship problem by examining a one-parameter family of related problems and then obtaining an analytic asymptotic approximation.

The Friedrichs Technique
First Expansion Procedure

We make a transformation of coordinates to a coordinate system rigidly fixed on the ship. After this transformation the fluids has velocity

$$(-U,\ 0,\ 0)$$

at infinity. We assume that the flow reaches a steady state so that the differential equations become

$$(\nabla \cdot \vec{q}) = 0 \tag{7}$$

$$\rho(\vec{q} \cdot \nabla)\vec{q} = -\nabla p + \rho g \epsilon_z \ . \tag{8}$$

The boundary conditions are:

$$(q \cdot n) = 0 \qquad \text{on ship} \tag{9}$$

$$\left.\begin{aligned} u\eta_x + v\eta_y - w &= 0 \\ p &= p_0 \end{aligned}\right\} \qquad \text{on free surface} \tag{10}$$

$$q \to (-U,\ 0,\ 0) \qquad \text{at infinity .} \tag{11}$$

In order to study this problem by means of the Friedrichs technique, we consider the above problem for a one-parameter family of ships S_b related to the ship we wish to study by a parameter b. The hull S_b described by the hull functions

$$-a^- \leq x \leq a^+$$

$$y = by^{\pm}\left[x,\ \frac{z}{b}\right] \tag{12}$$

$$-bz^-(x) \leq z \leq bz^+(x) \ .$$

These hulls are related to the hull given by Equation (1) by having similar cross sections in planes x = constant. When b tends to zero, the hulls shrink to a line. We consider this problem for values of b close to zero.

If the fluid is started from rest, the flow must be irrotational. Accordingly, a potential function Φ must exist with the property that

$$q = \nabla\Phi$$

The solution of the above one-parameter family of problems is a potential function $\Phi(x, y, z, b)$. We assume that in the limit as b tends to zero the flow must tend to the undisturbed flow described by the potential

$$\Phi(x, y, z, 0) = - Ux .$$

One can then write

$$\Phi(x, y, z, b) = - Ux + \phi(x, y, z, b) \quad , \qquad (13)$$

where ϕ is a function which vanishes in the limit as b tends to zero for a point x, y, z in the fluid.

The function Φ is the solution of the following problem:

$$\Delta\Phi = 0$$

$$\frac{\partial\Phi}{\partial n} = 0 \quad \text{on ship and free surface}$$

$$p = p_0 \quad \text{on free surface .}$$

Equation (9) may be integrated to obtain

$$\tfrac{1}{2}\rho q^2 + p - \rho g z = \text{constant}$$

Since at infinity $q = - U$, p is p_0, $\eta = 0$, we have

$$\tfrac{1}{2}\rho q^2 + p - \rho g z = \tfrac{1}{2}\rho\, U^2 + p_0$$

so that on the free surface,

$$\eta = \frac{1}{2g}(q^2 - U^2) \quad .$$

The normal direction to the ship S_b is in the direction of the vector

$$\left(b y_x^{\pm}, - 1, y_z^{\pm} \right) \quad .$$

Making use of Equation (15), we obtain the boundary conditions satisfied by ϕ :

On the ship's surface

$$\left(-U + \frac{\partial\phi}{\partial x}\right) by_x^{\pm} - \frac{\partial\phi}{\partial y} + y_z^{\pm} \frac{\partial\phi}{\partial z} = 0 \ .$$

The free surface shape is

$$\eta(x, y, b) = \frac{1}{2g}\left[-2(\nabla\phi \cdot U) + (\nabla\phi)^2\right] . \tag{14}$$

Using (10) we obtain

$$\left(- U + \frac{\partial\phi}{\partial x}\right) \eta_x + \frac{\partial\phi}{\partial y} \eta_y - \frac{\partial\phi}{\partial z} = 0 \tag{15}$$

on the free surface.

At infinity $\nabla\phi$ should tend to zero. Using Green's formula and the boundary conditions satisfied by ϕ, we can write the integrodifferential equation satisfied by the function $\phi(x, y, z, b)$. We obtain

$$\phi(x, y, z, b) = \frac{1}{4\pi} \int_{S_b} \left[\phi \left(\nabla \frac{1}{r} \cdot n\right) - \frac{1}{r}(U \cdot n) \right] dS$$

$$+ \frac{1}{4\pi} \int_{F_b} \left[\phi \left(\nabla \frac{1}{r} \cdot n\right) - \frac{1}{r}(U \cdot n) \right] dS \tag{16}$$

where

$$r^2 = (x - x')^2 + (y - y')^2 + (z - z')^2$$

with primed quantitied indicating integration variables and F_b the free surface corresponding to S_b.

We make the assumption that there exists a solution of our problem having uniformly bounded velocities and depending smoothly on the parameter b. Accordingly, $\phi(x, y, z, b)$ satisfies the integral Equation (16). If we let b tend to zero, we obtain:

$$\lim_{b \to 0} \phi(x, y, z, b) = \phi(x, y, z, 0) \quad ,$$

which must vanish. The free surface for the problem $b = 0$ is the surface

$$\eta(x, y, z, 0) = 0 \quad .$$

Since on this surface

$$(U \cdot n) = 0 \quad ,$$

this solution of the limit problem is in agreement with the limit of Equation (16) as b tends to zero.

THE ASYMPTOTIC BEHAVIOR OF THE SOLUTION OF SHIP PROBLEM TO ORDER b^2

We introduce the expansions

$$\phi(x, y, z, b) = \sum_{i=i}^{\infty} b^i \phi^i(x, y, z, b)$$

$$\eta = \sum_{i=1}^{\infty} b^i \eta_i$$

on ship and free surface

$$n = \sum_{i=0}^{\infty} b^i n_i$$

where $\phi^i(x, y, z, b)$, n_i, η_i are of order less than one in b.

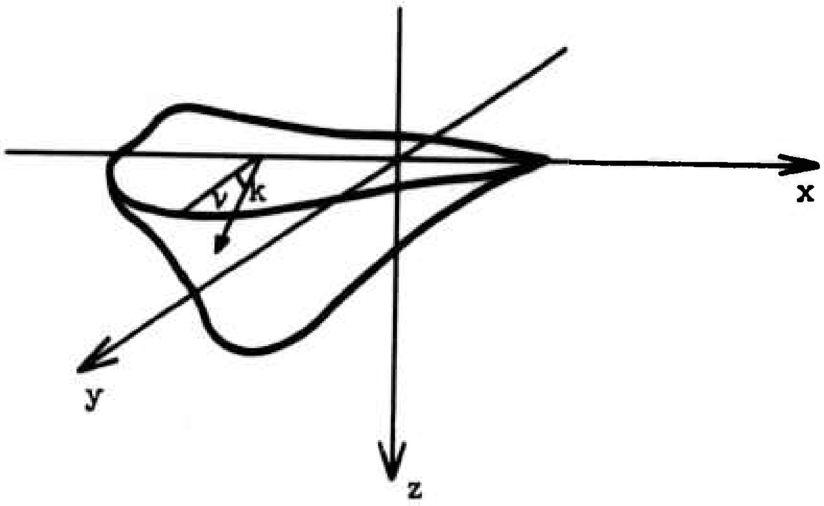


Figure 2

We note that if we describe the surface of S_1 by means of the cylindrical coordinate system described in Figure 2 we have

$$\theta^- \leq \theta \leq \theta^+$$

$$y = k(\theta,\ x)\cos\theta$$

$$z = k(\theta,\ x)\sin\theta$$

$$-a^- \leq x \leq a^+$$

and S_b is described by

$$\theta^- \leq \theta \leq \theta^+$$

$$y = bk(\theta,\ x)\cos\theta$$

$$z = bk(\theta,\ x)\sin\theta \qquad -a^- \leq x \leq a^+ \ .$$

We can then write

$$-\vec{n}dS = \begin{pmatrix} 0 \\ b(k_\theta \cos\theta - k\sin\theta) \\ b(k_\theta \sin\theta + k\cos\theta) \end{pmatrix} X \begin{pmatrix} 1 \\ bk_x \cos\theta \quad d\theta\ dx \\ bk_x \sin\theta \end{pmatrix}$$

so that

$$-\vec{n}dS = \begin{pmatrix} -b^2kk_x \\ b(k_\theta \sin\theta + k\cos\theta) \\ -b(k_\theta \cos\theta - k\sin\theta) \end{pmatrix} d\theta\ dx$$

Substituting the above expansions in Equation (25), we then obtain for the first order equation

$$\phi^1(x,\ y,\ z,\ b) = \frac{1}{4\pi}\int_{F_0}\left[\phi^1\left(\nabla\frac{1}{r}\cdot n_0\right) - \frac{1}{r}(U\cdot n_1)\right]dS \quad (17)$$

where F_0 is the plane $z = 0$.

The second-order equation is

$$\phi^2(x, y, z, b) = \frac{1}{4\pi} \int_{-a^-}^{a^+} \int_{\theta_0}^{\theta^+} \left[\phi^1 \left(\nabla \frac{1}{r} \cdot n_1 \right) - \frac{1}{r} Ukk_x \right] d\theta \; dx$$

$$+ \frac{1}{4\pi} \int_{F_0} \left[\phi^2 \left(\nabla \frac{1}{r} \cdot n_0 \right) - \frac{1}{r}(U \cdot n_2) \right] dS$$

$$+ \frac{1}{4\pi} \int_{F_0} \phi^1 \left[\left(\nabla \frac{1}{r} : n_1 \right) + \nabla \left(\nabla \frac{1}{r} \cdot n_0 \right) \cdot n_1 \right] dS.$$

Expanding Equation (24), we obtain for the terms of orders zero one and two

$$-U \cdot \eta_{0,x} = 0. \qquad \text{zero order} \qquad (18)$$

$$-U\eta_{1,x} + \frac{\partial \phi^1}{\partial x} \eta_{0,x} + \frac{\partial \phi^1}{\partial y} \eta_{0,y} - \frac{\partial \phi^1}{\partial z} = 0 \qquad \text{first order} \qquad (19)$$

$$-U\eta_{2,x} + \frac{\partial \phi^1}{\partial x} \eta_{1,x} + \frac{\partial \phi^2}{\partial x} \eta_{0,x} + \frac{\partial \phi^1}{\partial y} \eta_{1,y} + \frac{\partial \phi^2}{\partial y} \eta_{0,y} - \frac{\partial \phi^2}{\partial z} = 0 \qquad \text{second order.} \qquad (20)$$

Expanding Equation (14), we obtain

$$\eta_0 = 0 \qquad (21)$$

$$\eta_1(x, y, b) = - \frac{1}{g}(U \cdot \nabla\phi^1) \qquad (22)$$

$$+ \frac{1}{4\pi} \int_{F_0} \left[\phi^2 \left(\nabla \frac{1}{r} \cdot n_0 \right) - \frac{1}{r}(U \cdot n_2) \right] ds \qquad (24) \text{ cont'd}$$

with the boundary conditions

$$-U\eta_{2,x} - \frac{\partial \phi^2}{\partial z} = 0 \qquad (25)$$

and

$$\eta_2(x, y, b) = - \frac{1}{g} U \frac{\partial \phi^2}{\partial x} \qquad (26)$$

on the surface $z = 0$.

We let $S(x)$ represent the cross-sectional wetted area of the hull in the plane x = constant. Clearly

$$\int_{\theta^-}^{\theta^+} \tfrac{1}{2} k^2 d\theta = S(x) \quad ,$$

so that

$$\int_{\theta^-}^{\theta^+} k k_x d\theta + \tfrac{1}{2} k^2(\theta^+)\theta_x^+ - \tfrac{1}{2} k^2(\theta^-)\theta_x^- = S'(x) \quad . \qquad (27)$$

If we suppose that the wetted area on the ship occurs for $z \geq 0$, we obtain

$$0_x^+ = \theta_x^- = 0$$

so that we can write

$$\eta_2(x,\ y,\ b) = \frac{1}{2g}\left[\ -2(U \cdot \nabla\phi^2) - 2\nabla(U \cdot \nabla\phi^1) \cdot \eta_1 + (\nabla\phi^1)^2\right]. \quad (23)$$

Equation (18) is in agreement with Equation (21). Accordingly, Equation (19) becomes

$$-U\eta_{1,x} - \frac{\partial\phi^1}{\partial z} = 0 \ .$$

Equation (22) becomes

$$\eta_1(x,\ y,\ b) = -\frac{U}{g}\ \frac{\partial\phi^1}{\partial x} \ .$$

The function ϕ^1 is a harmonic function

$$\Delta\phi^1 = 0 \ .$$

and satisfies Equation (17). Clearly,

$$n_0 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

so that on letting $z > 0$ tend to zero in Equation (17), we obtain

$$\phi^1(x,\ y,\ 0,\ b) = -\frac{1}{2\pi}\int_{F_0} \frac{1}{r}(U \cdot n_1)\ \ dS \ .$$

Since ϕ^1 does not depend on the hull and is zero when a hull is not present, we conclude that $\phi^1 \equiv 0$ from which we have that $\eta^1 = 0$.

The second-order term $\phi^2(s,\ y,\ z,\ b)$ then satisfies the integral equation

$$\phi^2(x,\ y,\ z,\ b) = -\frac{1}{4\pi}\int_{-a^-}^{a^+}\int_{\theta^-}^{\theta^+} \frac{U}{r}\, kk_x\ d\theta\ dx \qquad (24)$$

$$\phi^2(s,\ y,\ z,\ b) = \frac{1}{4\pi}\int_{-a^-}^{a^+} \frac{U}{r}\, S'(x)\ dx$$

$$+ \frac{1}{4\pi}\int_{F_0}\left[\ \phi^2\left(\nabla\frac{1}{r}\cdot n_0\right) - \frac{1}{r}(U\cdot n_2)\right] dS \qquad (28)$$

Equation (28) evaluated at $z = 0$ by letting z tend to zero through positive values of z becomes

$$\phi^2(s,\ y,\ 0,\ b) = -\frac{1}{2\pi}\int_{-a^-}^{a^+} \frac{U}{r}\, S'(x)\ dx - \frac{1}{2\pi}\int_{F_0}\frac{1}{r}(U\eta_{2,x})\ dS \qquad (29)$$

with

$$\eta_{2,x} = -\left(\frac{1}{U}\frac{\partial\phi^2}{\partial z}\right)_{z=0} \qquad (30)$$

and

$$\eta_2 = -\left(\frac{U}{g}\frac{\partial\phi^2}{\partial x}\right)_{z=0} \qquad (31)$$

Equations (29), (30), and (31) form an integral equation for the function $\phi^2(x,\ y,\ 0,\ b)$ from which we can obtain η_2. Thus $\phi^2(x,\ y,\ z,\ b)$ is determined from relation (46) once $\phi^2(x,\ y,\ 0,\ b)$ and $\eta_{2,x}$ are determined.

Once method of solving the integral equation is to observe that $\phi^2(x,\ y,\ z,\ b)$ is a potential function, i.e., a solution of the partial differential equation

$$\Delta\phi^2(x,\ y,\ z,\ b) = 0 \qquad (32)$$

in the fluid region outside the hull S_b with the boundary conditions

$$\frac{b^2 \partial \phi^2}{\partial n} = (U \cdot n) \quad \text{on} \quad S_b \tag{33}$$

$$\frac{\partial \phi^2}{\partial z} = \frac{U^2}{g} \frac{\partial^2 \phi^2}{\partial x^2} \quad \text{on} \quad z = 0 \tag{34}$$

The Green's function G(x, y, z, x', y', z', b) for Equation (32) with boundary condition (34) can be written, Stoker,[7] Peters and Stoker[6]

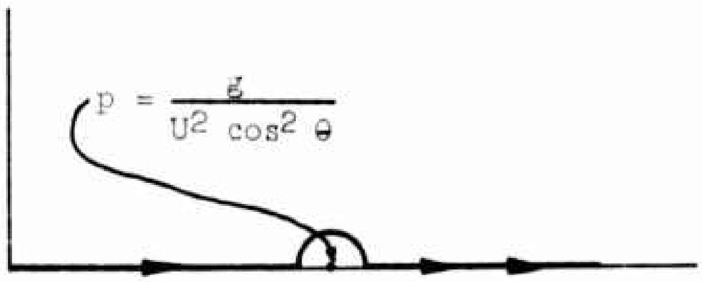
$$g(x,y,z,x',y',z') = \frac{1}{r(x-x',\, y-y',\, z-z')} - \frac{1}{r(x-x',\, y-y',\, z+z')} + \bar{G}$$

with

$$r(x-x',\, y-y',\, z-z') = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}\,,$$

$$\bar{G} = -\frac{4}{\pi} \operatorname{Re} \int_{\mathcal{P}} \int_0^{\pi/2} \frac{e^{-p(z-z')}}{\frac{U^2}{g} p \cos^2 \theta - 1} e^{i[p(x-x')\cos\theta]} \cos[p(y-y')\sin\theta]\, d\theta\, dp$$

where the path $\mathcal{P}$ is given by



p-plane

The path $\mathcal{P}$ is chosen so that $\bar{G} \to 0$ as x tends to $+\infty$.

Using this Greens function, we can then write by means of Greens theorem

$$\phi^2(x,y,z,b) = \frac{1}{4\pi}\int_{S_b} \phi^2 \frac{\partial G}{\partial n} - \frac{\partial \phi^2}{\partial n} G \quad dS$$

$$+ \frac{1}{4\pi}\int_{F_b} \phi^2 \frac{\partial G}{\partial n} - \frac{\partial \phi^2}{\partial n} G \quad dS$$

where n is the unit normal in the outward direction away from the fluid.

Letting $b \to 0$, we obtain

$$\phi^2(x,y,z,b) = -\frac{1}{4\pi}\int_{S_b} (U^{\infty} \cdot n) G \, dS$$

$$+ \frac{1}{4\pi}\int_{F_0} \left[\phi^2 \frac{\partial G}{\partial z} + U \cdot \eta_{2,x} G \right] dS \, .$$

Since

$$\eta_{2,x} = -\frac{U}{g}\frac{\partial^2 \phi^2}{\partial x^2}$$

one obtains

$$\int_{F_0} \phi^2 \frac{\partial G}{\partial z} - \frac{U^2}{g} G\phi^2_{xx} \quad dS = \int_{F_0} \frac{U^2}{g}\left[\phi^2 G_{xx} - G\phi^2_{xx}\right] dS =$$

$$= \int_{S_b F_0} \frac{U^2}{g} n_x \left(\phi^2 G_x - G\phi^2_x\right) \quad ds = O(b^3)$$

since ϕ^2 and ϕ_x^2 are of $O(b)$ and n_x is of $O(b^2)$. Here, $S_b F_0$ is the curve formed by the intersection of the plane $z = 0$ with the hull S_b .

The solution of our problem is therefore

$$\phi^2(x,y,z,b) = -\frac{1}{4\pi} \int_{S_b} (U \cdot n_2) G \, dS \tag{35}$$

In the limit as b tends to zero

$$-U \, n_{2,x} \, dS = -Ukk_x \, d\theta dx$$

so that

$$\phi^2(s,y,z,b) = -\frac{1}{4\pi} \int_{S_0} U \bar{G} S' dx \tag{36}$$

since the terms

$$\frac{1}{r(x-x', y-y', z-z')} - \frac{1}{r(x-x', y-y', z+z')}$$

tends to zero of order b as $z' = bk \sin \Theta$, tends to zero.

Using a similar procedure, the solution of the second order problem of two ship hulls situated on the lines $y = \pm d$, $z = 0$ with radii $k_+(\Theta,x)$ and $k_-(\Theta,x)$ is given by

$$\phi^2(x,y,z,b) = -\frac{b^2}{4\pi} \int_{S_{0,1}} U^\infty k_+ k_{+x} \bar{G}(x,x',y,d,z,z') \, dx d\theta$$

$$- \frac{b^2}{4\pi} \int_{S_{0,2}} U^\infty k_- k_{-x} \bar{G}(x,x',y,-d,z,z') \, dx d\theta \tag{37}$$

WAVE RESISTANCE

In this section, we calculate the wave resistance of the ship S_b. Bernoulli's integral of Equation (4) takes the form

$$\tfrac{1}{2}\rho q^2 + p - \rho g z = \text{constant}$$

Let $p = 0$ on the free surface. Since at $x = \infty$, $z = 0$, $p = 0$ on the free surface and $g = U$ we find that

$$\tfrac{1}{2}\rho q^2 + p - \rho g z = \tfrac{1}{2}\rho U^2 \quad .$$

We have obtained

$$\vec{q} = \vec{U} + b^2 \nabla \phi^2 + \ldots$$

so that

$$p = \rho g z - b^2 \rho U \nabla_x \phi^2 + \ldots$$

The wave-resistance R is then

$$R = \int_{S_b} p n_x \, dS \tag{38}$$

where the integral is taken over the wetted surface of S_b. The wetted surface can only be obtained after the free surface intersection with S_b is determined. This can not be done by the first expansion procedure without additional information. We thus assume that the intersection of the free surface with the ship is on $z = 0$.

Evaluating R and using the relation

$$n_x \, dS = b^2 k k_x \, d\theta \, dx$$

we have

$$R = -\rho U b^4 \int_{S_b} \nabla_x \phi^2 \, k k_x \, d\theta \, dx \quad .$$

We use the expression obtained previously for ϕ^2

$$\phi^2 = -\frac{1}{4\pi}\int_{S_0} U\, kk_x\, \bar{G}\, d\theta\, dx$$

with

$$\bar{G} = -\frac{4}{\pi}\,\mathrm{Re}\int_{\mathcal{C}}\int_0^{\pi/2} \frac{e^{-p(z-z')}}{\frac{U^2}{g}\,p\cos^2\theta - 1}\, e^{ip(x-x')\cos\theta}\cos p(y-y')$$

$$\times \sin\theta\, d\theta dp$$

The resistance R can be written in the form

$$R = \frac{\rho U^2 b^4}{\pi^2}\int_{S_b}(kk_{\bar{x}})\int_{S_0}(kk_x)\,\bar{G}_x\, d\theta\, dx\, d\bar{\theta}\, d\bar{x}$$

Since the contribution of the integration over the real axis in the expression for $\bar{G}$ is an integral involving $\sin(x-\bar{x})\cos\theta$ this contribution is zero because it changes sign when x and $\bar{x}$ are interchanged. The only contribution is that due to the path over the point $p = \frac{1}{\frac{U^2}{g}\cos^2\theta}$. The result is then

$$R = \frac{\rho U^2 b^4}{\pi}\int_{S_0}\int_{S_0}(kk_x)\,(kk_{\bar{x}})\int_0^{\pi/2}\frac{e^{-\frac{z}{\frac{U^2}{g}\cos^2\bar{\bar{\theta}}}}}{\frac{U^4}{g^2}\cos^3\bar{\bar{\theta}}}\cos\frac{x-\bar{x}}{\frac{U^2}{g}\cos\bar{\bar{\theta}}}$$

$$\cos\frac{y\sin\theta}{\frac{U^2}{g}\cos^2\bar{\bar{\theta}}}\quad d\bar{\bar{\theta}}dxd\theta d\bar{x}d\bar{\theta}$$

Integrating by parts with respect to x and $\bar{x}$ we obtain after setting $b = 0$ in the integrand

$$R = -\frac{\rho U^2 b^4}{\pi} \int\int (kk_x)_x (kk_{\bar{x}})_{\bar{x}} \left[\int_0^{\pi/2} \sec\bar{\bar{\theta}} \cos\frac{x-\bar{x}}{\frac{U^2}{g}\cos\bar{\bar{\theta}}}\, d\bar{\bar{\theta}}\right] d\theta\, dx\, d\bar{\theta}\, d\bar{x}$$

where we assume that at either end of the hull either k or k_x vanish.

Making use of the relation

$$\frac{1}{\pi}\int_0^{\pi/2} \sec\theta \cos\left(\frac{(x-\bar{x})}{\frac{U^2}{g}} \sec\theta\right) d\theta = -\frac{1}{2} Y_0\left(\frac{g}{U^2}(x-\bar{x})\right)$$

where Y_0 is Weber's zero order Bessel Function of the second kind, we obtain

$$R = -\frac{\rho U^2 b^4}{2} \int_{S_0}\int_{S_0} (kk_x)_x (kk_{\bar{x}})_{\bar{x}}\; Y_0\left(\frac{g}{U^2}(x-\bar{x})\right) dx\, d\bar{x}\, d\theta\, d\bar{\theta} \tag{39}$$

This is the same as the resistance found by Vossers[6] and Tuck.[7] We note that the only assumption made about k is that $kk_x = 0$ at the bow and stern.

In case the ship is made up of two hulls we have obtained the second order perturbation potential in Equation (37). Using this potential we then obtain

$$R_1 = \frac{\rho U^2 b^4}{2} \int_{S_1}\int_{S_1} (k_1 k_{1x})_x (k_1 k_{1\bar{x}})_{\bar{x}}\; Y_0\left(\frac{\rho g}{U^2}(x-\bar{x})\right) dx\, d\bar{x}\, d\theta\, d\bar{\theta}$$

$$+\ \frac{\rho U^2 b^4}{2\pi}\int_{S_1}\int_{S_2}(k_1 k_{1x})_x (k_2 k_{2\bar{x}})_{\bar{x}}\int_0^{\pi/2}\sec w\cos\left(\frac{pg}{U^2}\ (x-\bar{x})\ \sec w\right) X$$

$$X \cos\frac{2d}{\frac{U^2}{g}\cos^2 w}\,dw\,dx\,d\bar{x}\,d\theta\,d\bar{\theta}$$

$$+\ \frac{\rho U^2 b^4}{4\pi^2}\int_{S_1}\int_{S_2}(k_1 k_{1x})_x (k_2 k_{2\bar{x}})_{\bar{x}}\int_0^{\infty}\int_0^{\pi/2}\frac{\text{Sin }p(x-\bar{x})\cos w\cos 2pd\sin w}{\frac{U^2}{g}\,p\cos^2 w-1}$$

$$p\cos w\,dp\,dw\,dx\,d\bar{x}\,d\theta\,d\bar{\theta}$$

where R_1 is the resistance of the hull S_1; the expression for R_2 the resistance of the hull S_2, is similar.

The interference resistance is given by the last two terms in the expression for R. When the two hulls are the same and lie parallel to each other the interference resistance is

$$R_{int.} = \frac{\rho U^2 b^4}{\pi}\int_{S_1}\int_{S_2}(k_1 k_{1x})_x (k_2 k_{2\bar{x}})_{\bar{x}}\int_0^{\pi/2}\sec w\cos\left(\frac{\rho g}{U^2}\ (x-\bar{x})\sec w\right) X$$

$$X \cos\left(\frac{2dg}{U^2}\,\frac{\sin w}{\cos^2 w}\right) dw\,dx\,d\bar{x}\,d\theta\,d\bar{\theta}$$

since the other term must vanish. When the hulls are different or if the hulls are not parallel to each other there is in addition to a resistance a moment force tending to rotate the hull.

In this section we examine the nature of hull shapes which make the wave-resistance a minimum for a fixed displacement. This problem has been solved by Maruo.(3) However, we feel that this approach here is justified by its simplicity. We look at the wave-resistance R given by the relation

$$R = \frac{\rho U^2 b^4}{2} \int\int (kk_x)_x (kk_{\bar{x}})_{\bar{x}} \, Y_0 \left(\frac{g}{U^2} (x-\bar{x}) \right) d\theta d\bar{\theta} dx d\bar{x}$$

We take $a^+ = \ell$, $a^- = \ell$ and consider the behavior of R for large values of $\frac{U^2}{g\ell}$. We use the asymptotic formula

$$Y_0 (x) \approx J_0 (x) \log x + 0 (x^2) \quad \text{as } x \to 0$$

since

$$J_0 (x) \approx 1 + 0 (x^2)$$

we obtain

$$R \approx \frac{\rho U^2 b^4}{2} \int\int (kk_x)_x (kk_{\bar{x}})_{\bar{x}} \log \left[\frac{g\ell}{U^2} \frac{(x-\bar{x})}{\ell} \right] d\theta d\bar{\theta} dx d\bar{x}$$

$$+ 0 \left(\frac{g\ell}{U^2} \right)^2$$

so that, setting $S = \frac{1}{2} \int k^2 d\theta$, we have

$$R \approx \frac{2\rho U^2 b^4}{} \int\int S'' \bar{S}'' \log \left(\frac{g\ell}{U^2} \frac{(x-\bar{x})}{\ell} \right) dx \, d\bar{x} \qquad (40)$$

and for large values of $\frac{U^2}{g\ell}$ the problem becomes one of minimizing R with the condition that

$$\int S(x)\, dx = V$$

The above expression for R is similar to the expression obtained by von Karman for the wave drag of a slender cone.

Since $S' = 0$ at the bow and stern of the ship the variational problem results in the integral equation

$$\frac{d}{dx} \int \bar{S}'' (x-\bar{x})^{-1}\, d\bar{x} = \lambda$$

where λ is a Lagrange multiplier to be determined later. Integration with respect to x results in the integral equation

$$\int_{-\ell}^{+\ell} \bar{S}'' (x-\bar{x})^{-1}\, d\bar{x} = \lambda x + \mu$$

The solution of this integral equation is well known. It is, see Reference 8,

$$S''(x) = -\frac{1}{\pi^2} \int_{-\ell}^{+\ell} \frac{(\lambda y + \mu)}{x - y} \frac{\sqrt{\ell^2 - y^2}}{\sqrt{\ell^2 - x^2}}\, dy$$

Using the integral

$$\int_{-\ell}^{+\ell} \frac{\sqrt{\ell^2 - y^2}}{x - y}\, dy = \pi x$$

we can then write

$$S''(x) = \frac{\lambda}{\pi^2} \int \frac{\sqrt{\ell^2 - y^2}}{\sqrt{\ell^2 - x^2}}\, dy + \frac{\mu - \lambda x}{\pi^2 \sqrt{\ell^2 - x^2}} \int_{-\ell}^{+\ell} \frac{\sqrt{\ell^2 - y^2}}{x - y}\, dy$$

Since

$$\int_{-\ell}^{+\ell} \sqrt{\ell^2 - y^2}\, dy = \ell^2 \frac{\pi}{2}$$

one obtains

$$S''(x) = \frac{\lambda \ell^2}{2\pi \sqrt{\ell^2 - x^2}} + \frac{1}{\pi} \frac{\mu - \lambda x}{\sqrt{\ell^2 - x^2}}$$

Integrating the last relation, we obtain

$$S'(x) = \frac{\lambda x - 2\mu}{2\pi} \sqrt{\ell^2 - x^2}$$

and thus

$$S(x) = -\frac{\lambda}{6\pi} \left[\ell^2 - x^2\right]^{3/2} - \frac{\mu}{\pi}\left[\frac{\ell^2}{2} \text{arc sin} \frac{x}{\ell} + \frac{x}{2}\sqrt{\ell^2 - x^2}\right] + \frac{\mu \ell^2}{4} \quad (41)$$

If $S(-\ell) = 0$ then $\mu = 0$ and we obtain

$$S(x) = -\frac{\lambda}{6\pi} \left[\ell^2 - x^2\right]^{3/2}$$

Thus

$$V = \int S(x)\, dx = -\frac{\lambda \ell^4}{2}$$

and we obtain

$$\lambda = -\frac{2V}{\ell^4} \quad .$$

The wave-resistance is then

$$R = -4\rho \frac{U^2V^2}{\ell^4} b^4 \tag{42}$$

If $S(-\ell) \neq 0$ then

$$\mu = \frac{2S(-\ell)}{\ell^2}$$

and

$$V = -\frac{\lambda}{2}\ell^4 + \ell\, S(-\ell)$$

Therefore

$$\lambda = -2\,\frac{V - \ell\, S(-\ell)}{\ell^4}$$

The resistance R is

$$R = \left[\frac{4V\,S(-\ell)}{\ell^3} - \frac{2V^2}{\ell^4}\right] 2\rho U^2 b^4 \tag{43}$$

In the case of an axisymmetric hull we have

$$S(x) = \frac{\pi}{2} k^2(x)$$

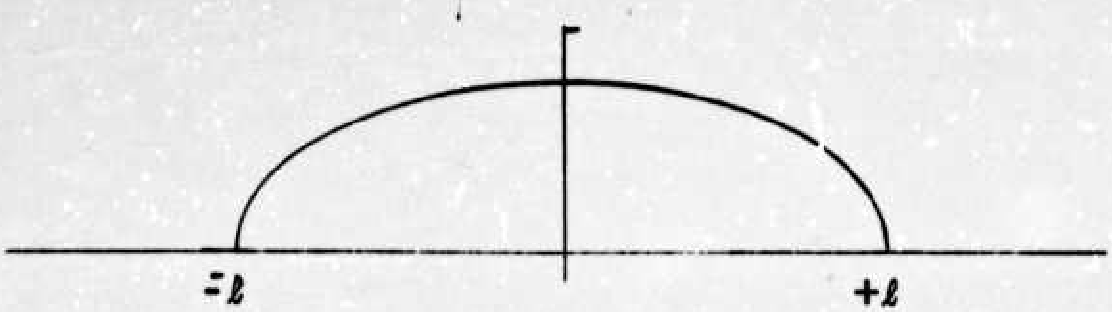
so that for an optimum hull pointed at both ends

$$\frac{\pi}{2} k^2(x) = \frac{2V}{6\pi\ell^4}\left[\ell^2 - x^2\right]^{3/2}$$

and

$$k(x) = \frac{2}{\pi\ell^2}\sqrt{\frac{V}{6}}\left[\ell^2 - x^2\right]^{3/4}$$

The area distribution for an optimum axisymmetric hull pointed at both ends is described in the following figure and we can see that at both ends the radius distribution is blunt.



CONCLUSIONS

We have used a perturbation procedure to obtain the solution of the ship problem to second order in the perturbation parameter. The wave-resistance of a ship moving with uniform velocity over a semi-infinite ocean is then evaluated and we find that for large values of the Froude number that the wave-resistance is similar to the wave-resistance of a slender body moving through a compressible gas. We then find for this case the hull shapes of constant displacement which make the resistance a minimum. In the case of an axisymmetric hull we find that the radius curve must be blunt.

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A NOTE ON SLENDER-BODY THEORY

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A NOTE ON SLENDER-BODY THEORY

In this note I shall be concerned with a version of slender-body theory for surface ships at zero forward speed already published by me in 1962, and with some of the interesting problems that still remain unsolved. We are concerned with the fluid motion near an oscillating ship and with the resulting hydrodynamical forces. In practice gravity effects (such as wave resistance) and viscous effects (such as skin friction) are both important and cannot be separated but since no existing theory includes both we here neglect viscous effects. The motion is then irrotational and is described by a velocity potential. Even with this simplification the boundary-value problem for surface ships is intractable unless the equations of motion can be linearized, for which purpose additional simplifying assumptions are made: only such bodies and ship motions are treated as will cause the resulting fluid motion to differ only slightly from a state of rest or of uniform flow.

For instance, we may consider a thin-ship model, in effect a vertical plate of small thickness moving horizontally along itself with uniform velocity (Michell 1898). Such a thin ship, of beam much smaller than draft and length, can be represented by a distribution of Kelvin wave sources over its mid-plane (for review see Wehausen 1957). The oscillations of the Michell ship have recently been studied by Peters and Stoker (1957) who have confirmed that in the first approximation its virtual mass and wave damping both vanish. The Michell model is therefore incomplete when virtual mass and wave damping are of interest, as in most problems of ship motion in waves.

To overcome this defect it is natural to consider slender ships, of comparable beam and draft, and of length much greater than either. Since evidently the disturbance tends to zero as beam and draft both tend to zero (when the ship contracts to a line), a scheme of linearization based on the thickness-length ratio is reasonable. A considerable amount of work has already been published on submerged slender ships, but it is generally realized that the motion near a submerged ship differs materially from the motion near a surface ship. Many of these calculations have been concerned with strip theories. It can be seen that near a slender body the motion in planes normal to the ship's axis is nearly plane irrotational and can, to a rough first approximation, be found without reference to the flow parallel to the axis. In the strip theories the three-dimensional motion is

synthesized approximately by combining these two-dimensional motions, and various ways of doing this have been suggested. Havelock (1956) has compared two strip methods of calculating the wave damping of a submerged spheroid at zero speed. The first method consists in calculating the two-dimensional energy transfer per unit length from a circular cylinder of radius equal to the local radius, and integrating this along the length of the cylinder; this clearly cannot give a good approximation when the waves are long compared to the length of the ship. In the second method the spheriod is replaced by an axial of the body, if the mean forward speed vanishes, and if the motion is (for simplicity) symmetrical about $\theta = 0$ and $\theta = 1/2\pi$, then in an unbounded medium the velocity potential is by this procedure found to have near the body the approximate form

$$\phi\,(x,\ r\cos\theta,\ r\sin\theta) = 2a_o(x)\ln\frac{L}{r} + 2\sum^{\infty}\frac{a_n\,\Gamma(x)}{2n}\,\frac{\cos 2n\,\theta}{r^{2n}} \tag{1.1}$$

$$-\int_{\infty}^{x}\frac{da_o(\xi)}{d\xi}\ln\left(\frac{L}{2|x-\xi|}\right)d\xi + \int_{x}^{\infty}\frac{da_o(\xi)}{d\xi}\ln\left(\frac{L}{2|x-\xi|}\right)d\xi \tag{1.2}$$

Full details are given in the 1962 paper. Arguments can be given to show that the expansions in terms of the wave sources and wave-free potential used there, though necessarily incomplete, are adequate for our purpose. For plane problems the expansions are complete (see Ursell, 1949 a, b), also for the oscillating sphere (Havelock 1955). The three-dimensional expansion used in the 1962 paper generalizes Havelock's work and was first proposed by Grim (1957, 1960) who did not however succeed in going beyond step (1) and could thus obtain neither an expansion analog to (1.1), (1.2) nor the interaction between sections. Grim observed that when this interaction is negligible the local source strength must be the same as for an infinitely long cylinder, of appropriate cross-section, and he suggested an iterative scheme of approximation.

A promising alternative treatment both for zero and non-zero speeds has recently been proposed by Vossers (1962). The linearized problem is by means of Green's theorem reduced to the solution of a linear Fredholm integral equation of the second kind for the potential

distribution of wave dipoles, of strength depending on the local area of cross-section, and this gives good results provided only that the spheroid is sufficiently slender. Both these methods have been applied to surface ships at zero speed, the first by Tasai (1959), the second by Grim (1957, 1960) who points out that for surface ships of finite length the first method wrongly predicts an infinite virtual mass for infinite wave-length.

It was the purpose of my 1962 paper to adapt for surface ships the techniques of the well-established aerodynamical slender-body theory for incompressible unbounded media (for an account see Thwaites 1960, Chap. 9, para. 11). One version of the theory for unbounded media proceeds in the following steps (presented here for a body of revolution):

(1) Approximate the body by axial line-distributions of known point singularities (sources and multipoles), whose strength is to be determined.

(2) By means of the Fourier convolution theorem express the velocity potentials of these line distributions in terms of the Fourier transforms (in the axial direction) of the point-singularity potentials.

(3) Expand these Fourier transforms in powers of the radius (logarithms will also appear), and retain only the leading terms. (It is here that the slender-body assumption is used.)

(4) By means of the Fourier convolution theorem interpret the resulting expressions.

This procedure can be justified if the body is sharply pointed at the ends, and can probably be modified to provide end corrections for blunt bodies (in an unbounded medium). If cylindrical polar coordinates (x, r, θ) are taken, with the x-axis along the axis on the ship on which the normal velocity is prescribed. If now the variation in the axial direction is treated as slow, then the integrals can be approximated by simpler integrals; in particular the kernel of the equation may be expected to reduce to the kernel appropriate to a plane problem, to which known techniques are applicable.

Vossers's results require approximations for quadruple integrals, of the same order of difficulty as those obtained above by the Fourier-transform techniques. Although his published treatment probably contains errors, it is believed that it can be modified to give at least the leading terms in the correct equations. Vossers's approach clearly has advantages; for instance it does not need to assume that the ship can be represented by axial distributions of multipoles. As for higher approximations, these are so far known only for zero speed and for bodies of revolution, so that except for this special case Vossers's method may in this respect be as efficient as other methods.

My own theoretical work was originally inspired by some experiments on the oscillations of an idealized model ship moving through waves. Resonances were observed at certain periods of encounter which agreed quite well with the periods predicted by thin-ship theory. But the amplitudes could not be explained by this theory which does not take virtual mass and wave damping into account, and slender-body theory seemed likely to give additional information on these. However, the mass of a very slender body is so small that resonances can never occur. It may be that thin-ship and slender-ship theory can be combined into a linear theory (perhaps by Vossers's approach) to explain such observations, or it may be that non-linear terms are also needed. Such questions still require investigation.

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AN APPROACH TO THIN-SHIP THEORY

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AN APPROACH TO THIN-SHIP THEORY

Introduction

In the usual derivation of the thin-ship approximation of the velocity potential for a ship in steady or unsteady motion one begins with the exact boundary conditions for the problem, assuming irrotational flow of an inviscid fluid. The various unknown functions are then assumed to be analytic functions of some parameter describing the thinness of the ship. After various manipulations which need not be repeated here, one has replaced the original boundary-value problem by a new linearized one whose solution is taken to be an approximation of the desired one. However, there is always some question as to the proper restrictions for this approximation to be a good one. The question has recently been broached by Maruo (1962) who states that the beam must be small in comparison with not only the length but also the draft and c^2/g. Although his statements are not supported by arguments (in the paper), there is no question but that examination of the terms discarded do no leave one with a clear impression that only small derivatives in the x-direction are necessary. In particular, some of the discarded terms contain derivatives in a vertical direction which may be infinite at the keel of a U-sectioned ship.

The purpose of this paper is to present a derivation of the thin-ship velocity potential which may make more evident the proper conditions for its use as an approximation to the exact one.

The problem will be formulated sufficiently generally so that both steady motion and motion started from rest are included in the equations. However, the two cases will be treated separately thereafter.

Exact Formulation of the Problem

It will be convenient to introduce three right-handed coordinate systems, one fixed in space $\bar{O}\,\bar{x}\,\bar{y}\,\bar{z}$, one fixed in the ship $O'\,x'\,y'\,z'$ and one moving with the ship but parallel to the space coordinates $O\,x\,y\,z$. We take $\bar{O}\,\bar{y}$ directed oppositely to the force of gravity, $\bar{O}\,\bar{x}$ to coincide with the direction of motion, and $\bar{O}\,\bar{x}\,\bar{z}$ to coincide with the undisturbed water surface. Further,

$0\ x\ y\ z$ and $0'x'y'z'$ coincide when the ship is at rest. $0'x'y'$ contains the centerplane section of the ship and $0'y'z'$ and midship section. Let $\bar{0}0 = \bar{x}\ (t)$, where $\dot{\bar{x}}\ (t) = c(t)$.

Let the center of gravity of the ship be at the point $(x_G', y_G', 0)$, its mass be m, and its pitching moment of inertia about the center of gravity be I_z. The hull will be described by

$$z' = \pm f(x',y'). \qquad (1)$$

Let $L = 2\ell$ be the length at the undisturbed water surface, $B = 2b$ the beam amidships and H the draft there. By the "profile" of the ship we shall mean the intersection of the hull with the plane $0'x'y'$. We assume that $f(x',y')$ vanishes on the profile. Although one could allow more generality in this respect, it is obvious that, if the ship's bottom is flat, a minute change in the form will allow this condition to be fulfilled. On the other hand, a dished-in bottom could not be included in the description (1). The function f will be assumed to have continuous derivatives with respect to each variable.

We shall suppose that the motion of the ship results from a thrust T acting parallel to $0'x'$, and applied along a line situated at a distance d below the center of gravity. When the ship is moving, both its trim and the position of its center of gravity relative to $0\ x\ y\ z$ will change. Let α be the trim angle, measured positively in the bow-up direction and let e be the amount by which the origin $0'$ is raised (see Figure 1). Then $0\ x\ y\ z$ and $0'x'y'z'$ are related by the equations

$$\begin{aligned} x' &= x \cos\alpha + (y-e)\sin\alpha, & x &= x'\cos\alpha - y'\sin\alpha, \\ y' &= -x\sin\alpha + (y-e)\cos\alpha, & y &= e + x'\sin\alpha + y'\cos\alpha, \\ z' &= z\ ; & z &= z'. \qquad (2) \end{aligned}$$

The motion of the fluid is most easily described in the systems $0\ \bar{x}\ \bar{y}\ \bar{z}$ or $0\ x\ y\ z$. Let $\phi(\bar{x},\bar{y},\bar{z},t)$ be the velocity potential in the fixed system and $\varphi(x,y,z,t)$ that in the moving system. Then

$$\phi(\bar{x},\bar{y},\bar{z},t) = \varphi(\bar{x} - \bar{x}\ (t),\bar{y},\bar{z},t) \qquad (3)$$

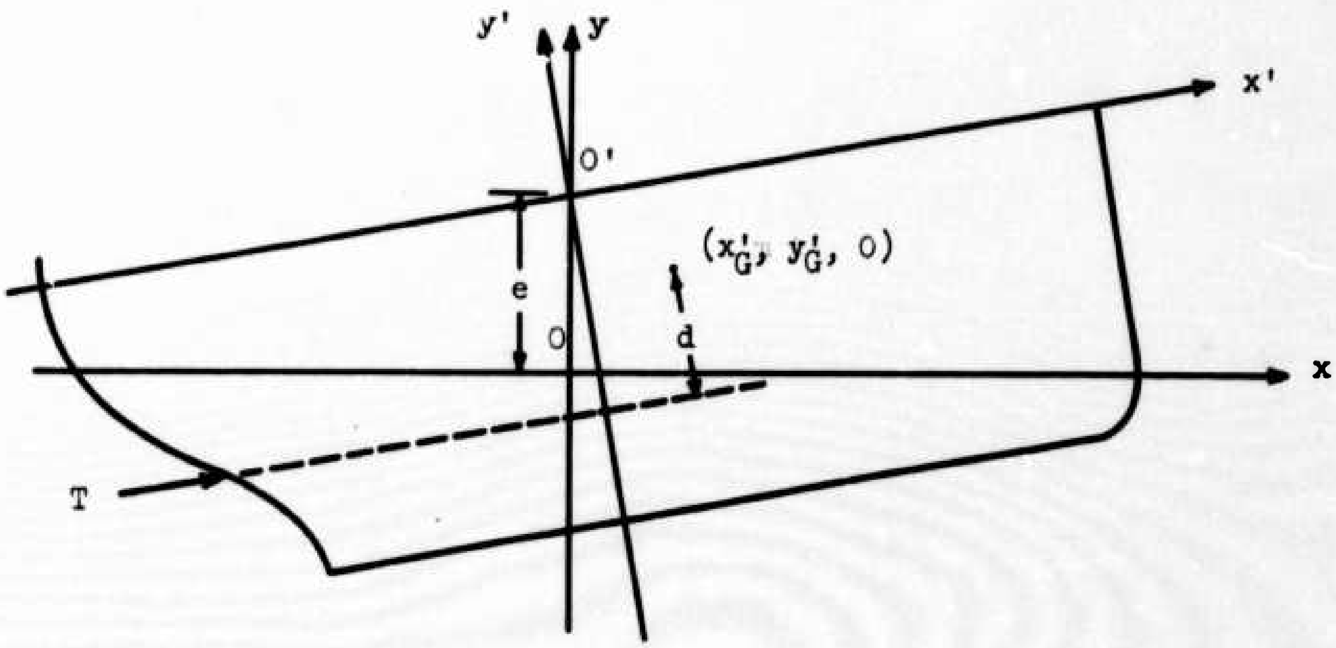

Figure 1

and

$$\phi_t = -c\varphi_x + \varphi_t, \quad \varphi_{\bar{x}} = \varphi_x \ , \quad \phi_{\bar{y}} = \varphi_y \ , \quad \phi_{\bar{z}} = \varphi_z \ .$$

If the motion is steady with respect to the ship, then $\varphi_t = 0$. Both grad ϕ and grad φ describe the absolute velocity of the water. The Bernoulli integral may be written as either one of

$$\phi_t + gy + \frac{1}{2}(\phi_{\bar{x}}^2 + \phi_{\bar{y}}^2 + \phi_{\bar{z}}^2) + \frac{p}{\rho} = \frac{p_a}{\rho} \ ,$$

$$\varphi_t - c\varphi_x + gy + \frac{1}{2}(\varphi_x^2 + \varphi_y^2 + \varphi_z^2) + \frac{p}{\rho} = \frac{p_a}{\rho} \tag{4}$$

where p_a is atmospheric pressure. We may evidently take $p_a = 0$ since we are not concerned with cavitation.

Let the equation of the water surface be

$$y = Y(x,z,t) = Y(\bar{x}-\bar{x}_o(t), \bar{z}, t) = \bar{Y}(\bar{x}, \bar{z}, t). \tag{5}$$

Then the potential function φ must satisfy the following boundary conditions on this surface:

$$\varphi_x(x,Y(x,z,t),z,t)\ Y_x(x,z,t) - \varphi_y + \varphi_z Y_z + Y_t - cY_x = 0,$$

$$\varphi_t(x,Y(x,z,t),z,t) - c\varphi_x + gY(x,z,t) + \frac{1}{2}(\varphi_x^2 + \varphi_y^2 + \varphi_z^2) = 0. \tag{6}$$

There are corresponding kinematic and dynamic boundary conditions to be satisfied on the ship's wetted surface. The kinematic condition is

$$\varphi_x(x,y,\pm f(x',y'),t)\frac{\partial}{\partial x} f(x',y') + \varphi_y \frac{\partial}{\partial y} f(x',y') \mp \varphi_z$$

$$= cf_{x'}\cos\alpha - cf_{y'}\sin\alpha - \frac{\partial}{\partial t} f(x',y'). \tag{7}$$

The last term on the right is necessary because both x' and y' depend upon t through α and e, both of which may vary in an initial-value problem. In the steady-state case this term vanishes. Form (2)

$$\begin{aligned} \frac{\partial}{\partial x} f(x',y') &= f_{x'}\cos\alpha - f_{y'}\sin\alpha , \\ \frac{\partial}{\partial y} f(x',y') &= f_{x'}\sin\alpha + f_{y'}\cos\alpha . \end{aligned} \tag{8}$$

In order to write down the dynamic boundary conditions, we first find the force and moment acting on the ship, assuming that we know the pressure $p(x,y,z,t)$, which can be found from (4) once φ is known. Denote the wetted hull by S_w and the projection of the wetted hull onto the centerplane by S_{wp}. The latter is bounded by the profile and the projection of the intersection of the water surface with the ship's surface. We shall denote the force components referred to 0 x y z by F_x, F_y and to 0'x'y'z' by F_x', F_y'. The moment about the center of gravity is denoted by M. Then

$$\begin{aligned} F_x' &= \iint_{S_w} p(x,y,z,t)n_x' dS = 2\iint_{S_{wp}} p(x,y,f(x',y'),t) f_{x'} dx'dy', \\ F_y' &= 2\iint_{S_{wp}} p(x,y,f(x',y'),t)\ f_{y'} dx'dy' , \\ M &= 2\iint_{S_{wp}} p(x,y,f,t)\ [(x'-x_G')f_{y'} - (y'-y_G')f_{x'}]dx'dy'. \end{aligned} \tag{9}$$

The dynamic boundary conditions are, of course, nothing but Newton's laws of motion:

$$
\begin{aligned}
m\ddot{\bar{x}}_G &= m[\ddot{\bar{x}}_o - (x_G' \sin\alpha + y_G' \cos\alpha)\ddot{\alpha} - (x_G'\cos\alpha - y_G'\sin\alpha)\dot{\alpha}^2] \\
&= F_x + T\cos\alpha = F_x' \cos\alpha - F_y' \sin\alpha + T\cos\alpha, \\
m\ddot{\bar{y}}_G &= m[\ddot{e} + (x_G' \cos\alpha - y_G' \sin\alpha)\ddot{\alpha} - (x_G'\sin\alpha + y_G'\cos\alpha)\dot{\alpha}^2] \\
&= F_y + T\sin\alpha - mg = F_x'\sin\alpha + F_y'\cos\alpha + T\sin\alpha - mg, \\
I_z\ddot{\alpha} &= M + Td .
\end{aligned}
\tag{10}
$$

For steady motion these reduce to

$$
\begin{aligned}
F_x + T\cos\alpha &= 0 , \\
F_y + T\sin\alpha - mg &= 0, \\
M + Td &= 0.
\end{aligned}
\tag{11}
$$

There is also a kinematic condition to be satisfied on the ocean bottom, which we assume to be a horizontal plane at $y = -h$:

$$\varphi_y(x, -h, z, t) = 0 \tag{12}$$

If the depth is infinite, this may be replaced by

$$\lim_{y \to -\infty} \varphi_y = 0. \tag{12'}$$

If there are walls parallel to the direction of motion, further conditions analogous to (12) must be added.

Finally there are conditions at infinity. Here we separate the steady from the unsteady case. For the initial-value problem we assume

$$\varphi(x,y,z,t) = O([x^2 + z^2]^{-1/2}) \text{ as } x^2 + z^2 \to \infty \tag{13}$$

For steady motion we assume

$$\varphi(x,y,z) = \begin{cases} O([x^2+z^2]^{-1/2}) & \text{as } x^2+z^2 \to \infty \text{ for } x>0, \\ O(1) & \text{as } x^2+z^2 \to \infty \text{ for } x<0. \end{cases} \qquad (14)$$

This problem can be modified in various ways to conform to the physical situation. For example, if a ship model is held rigidly by a towing carriage in its equilibrium position at rest, then $\alpha = e = 0$ and T is no longer applied as assumed above. The necessary modifications are easy to make, and the problem becomes, in fact, somewhat easier.

Steady Motion

For steady motion $\varphi = \varphi(x,y,z)$ and $Y = Y(x,z)$ and α, e and c are constants, so that in the boundary conditions for φ in (6) and (7) the partial derivatives with respect to t vanish. With this simplification, we are seeking a solution of Laplace's equation satisfying (6), (7), (11), (12) and (14).

Consider now the region of fluid bounded by the bottom S_B, the free surface S_F, the wetted hull S_W, and a vertical cylinder of radius R with axis Oy, S_R (see Figure 2).

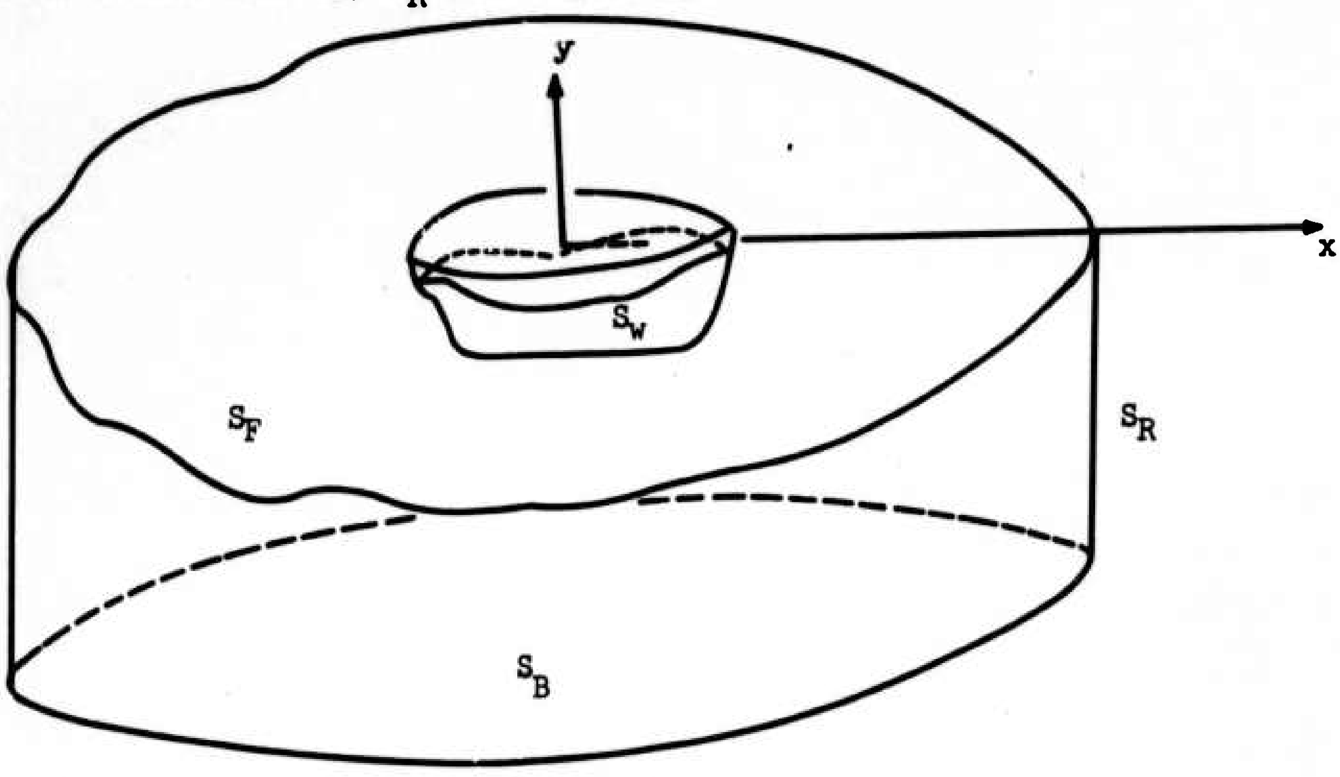

Figure 2

Let (x, y, z) be an arbitrary point of the region and (ξ, η, ζ) any point of the boundary. Further, let $G(x, y, z; \xi, \eta, \zeta)$ be a Green's function, that is, a solution of Laplace's equation in the variables ξ, η, ζ with a singularity like $1/r = [(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2]^{1/2}$ at (x, y, z) but otherwise harmonic in ξ, η, ζ, in the region occupied by fluid. Thus

$$G(x, y, z; \xi, \eta, \zeta) = \frac{1}{r} + H(x, y, z; \xi, \eta, \zeta), \qquad (15)$$

where H is harmonic. Further properties of G will be specified as they become desirable. We note that G is at our disposal in this respect, provided we are able eventually to construct it. Occasionally it will simplify the writing to replace x, y, z by P and ξ, η, ζ by Q, i.e. to write $G(P;Q)$. A subscript n will denote the normal derivative

$$G_n(P;Q) = G_x n_1 + G_y n_2 + G_z n_3$$

and a subscript ν the normal derivative

$$G_\nu(P;Q) = G_\xi \nu_1 + G_\eta \nu_2 + G_\zeta \nu_3.$$

Then by Green's Theorem we have the following formula:

$$\begin{aligned} \varphi(x, y, z) &= \frac{1}{4\pi} \int_S\int [\varphi_\nu(Q) G(P;Q) - \varphi(Q) G_\nu(P;Q)] dS(S) \\ &= \frac{1}{4\pi} \int\int_{S_R} [\varphi_R G - \varphi G_R] \, dS \\ &\quad - \frac{1}{4\pi} \int\int_{S_B} \varphi G_\eta \, dS \qquad (16) \\ &\quad + \frac{1}{4\pi} \int\int_{S_F} [\varphi_\nu G - \varphi G_\nu] dS + \frac{1}{4\pi} \int\int_{S_W} [\varphi_\nu G - \varphi G_\nu] dS \end{aligned}$$

Here we have already made use of (12) in the integral over S_B (really over that part of S_B inside the cylinder; similarly for S_F). We shall immediately add the following restriction to G:

$$G_\eta(x, y, z; \xi, -h, \zeta) = 0 \qquad (17)$$

If the depth is infinite, this is replaced by

$$\lim_{\eta \to \infty} G_\eta = 0 \qquad (17')$$

Then the integral over S_B vanishes.

Consider next the integral over S_F. Here

$$\vec{\nu} = (-Y_\xi, 1, -Y_\zeta)/[1 + Y_\xi^2 + Y_\zeta^2]^{1/2},$$

$$dS(Q) = [1 + Y_\xi^2 + Y_\zeta^2]^{1/2}. \qquad (18)$$

Denote the projection upon the (x,z)-plane of the part of S_F bounded by the hull and the cylinder by S_{FP}. We may then write this integral as follows:

$$\frac{1}{4\pi} \int\int_{S_{FP}} \{[-Y(\xi,\zeta)\varphi_\xi(\xi,Y(\xi,\zeta),\zeta) + \varphi_\eta - Y_\zeta\varphi_\zeta]G(x,y,z;\xi,Y,\zeta)$$

$$-[-Y_\xi G_\xi + G_\eta - Y_\zeta G_\zeta]\varphi\} \, d\xi d\zeta$$

From (6) this equals

$$\frac{1}{4\pi} \int\int_{S_{FP}} \{ -cY_\xi(\xi,\zeta)\, G(x,y,z;\xi,Y,\zeta) - G_\eta\varphi + [Y_\xi G_\xi + Y_\zeta G_\zeta]\varphi\} \, d\xi d\zeta$$

We recall that the formula for integration by parts in two dimensions is given by

$$\int\int_S F \operatorname{grad} G \, dS = \oint_C FG\vec{n} \, ds - \int_S\int G \operatorname{grad} F \, dS, \qquad (19)$$

where S is some region, C is its contour, $\vec{n}$ is exterior and orientation of C is counter-clockwise. Applying this to the first term above, we obtain

$$\frac{1}{4\pi} \oint - c\, YG\, n_1 \, ds$$

$$+ \frac{1}{4\pi} \int\int_{S_{FP}} \{ c\, Y(\xi,\zeta)\, [G_\xi + G_\eta Y_\xi] - G_\eta\varphi + [Y_\xi G_\xi + Y_\zeta G_\zeta]\varphi\} \, d\xi d\zeta.$$

From the second equation in (6) this equals

$$\frac{1}{4\pi} \oint - c\ YGn_1 ds + \frac{1}{4\pi} \int\int_{S_{FP}} \{c^2g^{-1}\ \varphi\ (\xi, Y(\xi, Y(\xi,\zeta),\zeta)G_\xi - G_\eta\varphi$$

$$+ c\ G_\eta YY_\xi + [Y_\xi G_\xi + Y_\zeta G_\zeta]\varphi - \frac{1}{2} c^2 g^{-1}\ [\varphi_\xi + \varphi_\eta + \varphi_\zeta]G_\xi\}\ d\xi d\zeta$$

After the $c^2g^{-1}\ \varphi_\xi G_\xi$ term interpolate $c^2g^{-1}\ \varphi_\eta Y_\xi G_\xi - c^2g^{-1}\varphi_\eta Y_\xi G_\xi$ so that the first two terms become

$$c^2g^{-1}\ G\ \frac{\partial}{\partial\xi}\ \varphi(\xi,\ Y(\xi,\zeta),\ \zeta).$$

After integrating this term by parts, we have

$$\frac{1}{4\pi} \oint [-c\ YG + c^2g^{-1}\ \varphi G_\xi]n\ ds - \frac{1}{4\pi} \int\int_{S_{FP}} [c^2g^{-1}\ G_{\xi\xi}(x,y,z;\xi,Y,\zeta)$$

$$+ G_\eta]\varphi(\xi,Y,\zeta)\ d\xi d\zeta\ \ \frac{1}{4\pi} \int\int_{S_{FP}} \{- c^2g^{-1}[\varphi G_{\xi\eta} + {}_\eta G_\xi]Y_\xi + c\ G_\eta YY_\xi$$

$$+ [Y_\xi G_\xi + Y_\zeta G_\zeta]\varphi - \frac{1}{2} c^2g^{-1}\ [\varphi_\xi^2 + \varphi_\eta^2 + \varphi_\zeta^2]\ G_\xi\}\ d\xi d\zeta \qquad (20)$$

For the time being we shall leave this integral in this form. The contour integral is taken about the contour shown in Figure 3.

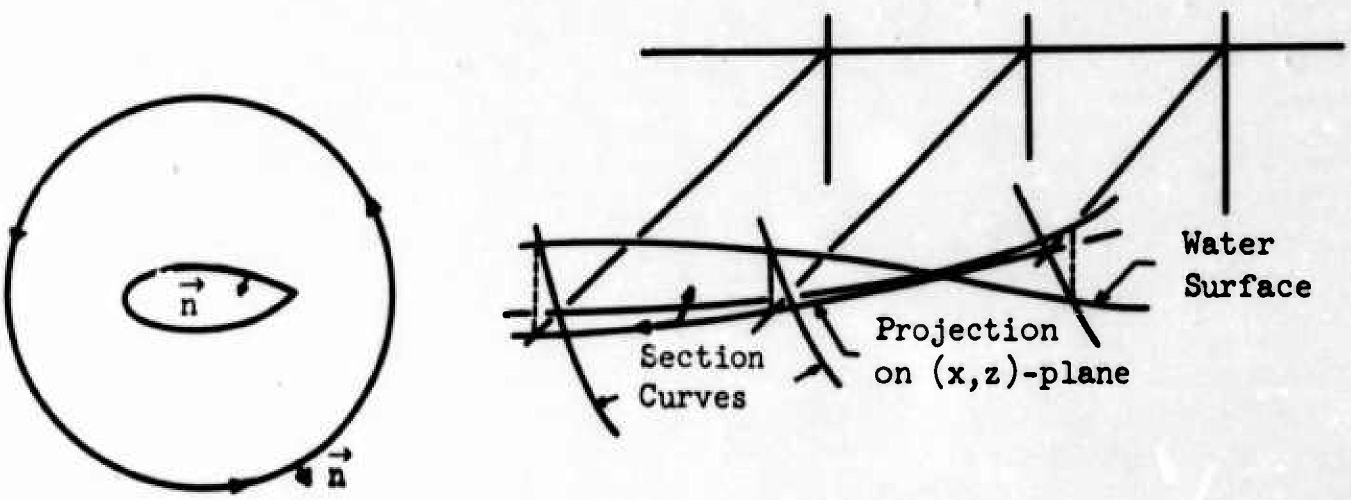

Figure 3

The points on the curve of intersection of the water surface with the hull must satisfy the equation

$$z = f(x', Y'(x,z)) = f(x \cos\alpha + [Y(x,z)-e] \sin\alpha, \; - x \sin\alpha + [Y-e] \cos\alpha)$$

for $z \geq 0$ and the same equation with z replaced by $-z$ for $z \leq 0$. We denote the explicit representation by $z = \pm \hat{f}(x)$. The normal vector for $z \geq 0$ is then given by

$$\vec{n}^{+} = (f'(x), \; -1)/ \; [1 + \hat{f}'^2]^{1/2}$$

For $z \leq 0$ it is given by

$$\vec{n}^{-} = (\hat{f}'(x), \; 1) \; [1 + f'^2]^{1/2}$$

Taking account of the direction of integration about the contour, we have

$$n_1 ds = - \frac{\hat{f}'}{-1} \, dx = \hat{f}'(x) \, dx \qquad \text{for} \qquad z \geq 0 \, ,$$

$$n_1 ds = \frac{\hat{f}'}{1} \, dx = \hat{f}'(x) \, dx \qquad \text{for} \qquad z \leq 0.$$

Making further use of the symmetry of both φ and Y with respect to z, we may write the part of the contour integral about the ship as follows:

$$\frac{1}{4\pi} \int \{-c \; [G(x,y,z;\xi,\hat{Y},\hat{f}) + G(x,y,z;\xi,\hat{Y},-\hat{f})] \; \hat{Y}$$

$$+ \; c^2 g^{-1} \; [G_\xi(x,y,z;\xi,\hat{Y},\hat{f}) + G_\xi(x,y,z;\xi,\hat{Y},-\hat{f}]\varphi(\xi,Y,f)\} d\xi \tag{21}$$

where $\hat{Y}(\xi) = Y(\xi,\hat{f}(\xi))$. We shall consider the other part of the contour later.

Consider now the integral over S_W. This may be calculated using either the (x',y') or the (x,y) system of coordinates. The form of the integrand will be the same in either case. From the boundary

condition (7) we have

$$\varphi_\nu(\xi,\eta,f(\xi',\eta')) = c(f_{\xi'}\cos\alpha - f_{\eta'}\sin\alpha).$$

The coordinates of n in the (x',y') system are

$$\vec{n}' = (f_{x'}, f_{y'}, \mp 1)/[1 + f_{x'}^2 + f_{y'}^2]^{1/2}$$

and in the (x,y) system

$$\vec{n} = (f_{x'}\cos\alpha - f_{y'}\sin\alpha,\ f_{x'}\sin\alpha$$
$$+ f_{y'}\cos\alpha,\ \mp 1)/[1 + f_{x'}^2 + f_{y'}^2]^{1/2} .$$

The element of area is the same for both sides of the ship:

$$dS = [1 + f_{x'} + f_{y'}]^{1/2}dxdy = [1 + f_{x'} + f_{y'}]^{1/2}dx'dy' .$$

The integral is then the following

$$\frac{1}{4\pi} \iint_{S_{WP}} \{c[f_{\xi'}(\xi',\eta')\cos\alpha - f_{\eta'}\sin\alpha][G(P;\xi,\eta,f(\xi',\eta'))$$
$$+ G(P;\xi,\eta,-f)] - \varphi(\xi,\eta,f(\xi',\eta'))[f_{\xi'}\cos\alpha$$
$$- f_{\eta'}\sin\alpha][G_\xi(P;\xi,\eta,f) + G_\xi(P;\xi,\eta,-f)]$$
$$- \varphi(\xi,\eta,f)[f_{\xi'}\sin\alpha + f_{\eta'}\cos\alpha][G_\eta(P;\xi,\eta,f)$$
$$+ G_\eta(P;\xi,\eta,-f)] + \varphi(\xi,\eta,f)[G_\zeta(P;\xi,\eta,f)$$
$$- G_\zeta(P;\xi,\eta,-f)]\}\, d\xi'dy' . \qquad (22)$$

In writing out the integral use has been made of the symmetry of φ, i.e.

$$\varphi(x,y,-z) = \varphi(x,y,z) .$$

An integration by parts in any of the first three terms replaces derivatives of f by f itself, but brings the derivatives back in again because of the occurrence of f as an argument in φ and G. We do this integration only for the first term and obtain the following expression:

$$\frac{1}{4\pi} \oint c\, f(\xi',\eta')[G(P;\xi,\eta,f) + G(P;\xi,\eta,-f)]\, n_1 ds$$

$$+ \frac{1}{4\pi} \iint_{S_{wP}} \{- c\, f(\xi',\eta')[G_\xi(P;\xi,\eta,f) + G_\xi(P;\xi,\eta,-f)]$$

$$- c\, f(\xi',\eta')[f_{\xi'}\cos\alpha - f_{1'}\sin\alpha][G_\zeta(P;\xi,\eta,f) - G_\zeta(P;\xi,\eta,-f)]$$

$$- \varphi(\xi,\eta,f)[f_{\xi'}\cos\alpha - f_{\eta'}\sin\alpha][G_\xi(P;\xi,\eta,f) + G_\xi(P;\xi,\eta,-f)]$$

$$- \varphi(\xi,\eta,f)[f_{\xi'}\sin\alpha + f_{\eta'}\cos\alpha][G_\eta(P;\xi,\eta,f) + G_\eta(P;\xi,\eta,-f)]$$

$$+ (\xi,\eta,f)[G_\zeta(P;\xi,\eta,f) - G_\zeta(P;\xi,\eta,-f)]\}\, d\xi' d\eta' . \qquad (23)$$

The contour integral is over the boundary of S_{wP}. However, since f vanishes on the profile of the ship, it is taken in effect only along the projection of the wave profile onto the centerplane (see Figure 4). The components of $\vec{n}$ in this formula are those in the (x,y) system. The points of the projected wave profile must satisfy

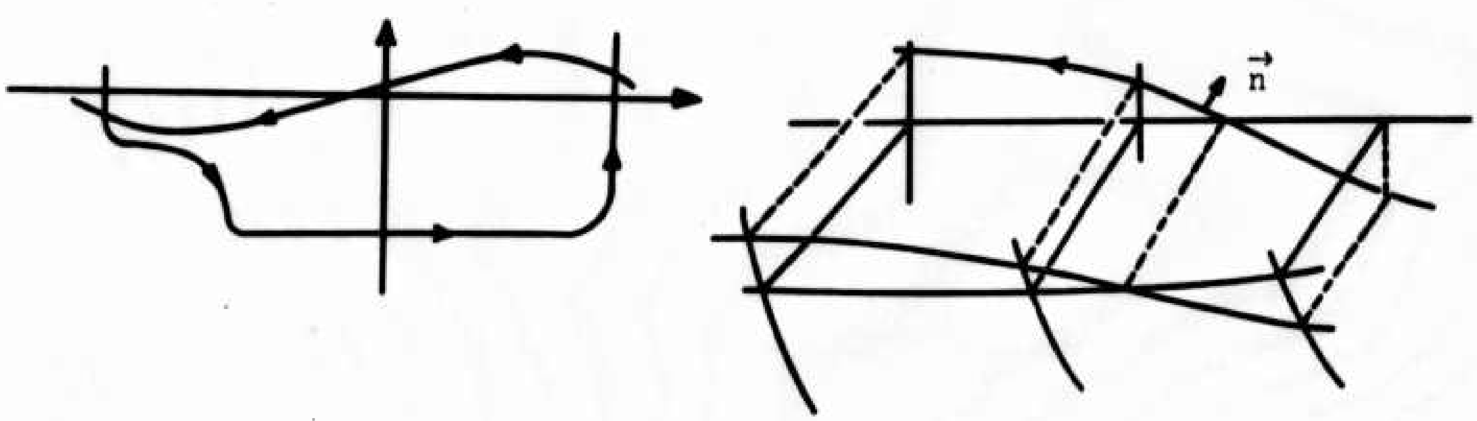

Figure 4

$y = Y(x, f(x', y') = Y(x, f(x \cos\alpha + (y-e) \sin\alpha, -x \sin\alpha + (y-4) \cos\alpha);$
if solved for y, say $y = \hat{Y}(x)$, this represents the curve explicityly. The normal vector is given by

$$\vec{n} = (-\hat{Y}'(x), 1)/[1 + \hat{Y}'^2]^{1/2} \tag{24}$$

Since here

$$n_1 ds = \hat{Y}'(x)dx, \quad n_1 ds = dx , \tag{25}$$

the contour integrals can be written out explicitly in terms of f, Y, φ and G. The contour integral in (23) may be combined with that in (21) to give

$$\begin{aligned} &\frac{1}{4\pi} \int c \,[G(P;\xi,\hat{Y}(\xi),\hat{f}(\xi)) + G(P;\xi,\hat{Y},-\hat{f})][\hat{f}\hat{Y}' - \hat{f}'\hat{Y}]\, d\xi \\ &+ \frac{1}{4\pi} \int c^2 g^{-1} [G_\xi(P;\xi,Y,f) + G_\xi(P;\xi,Y,-f)]\,(\xi,Y,f)f' d\xi \,. \end{aligned} \tag{26}$$

Lastly we turn to the integral around the vertical cylinder S_R:

$$\frac{1}{4\pi} \iint_{S_R} [\varphi_R G - \varphi G_R] dS = \frac{R}{4\pi} \int_0^{2\pi} d\varphi \int_{-h}^{Y} [\varphi_R G - \varphi G_R]\, dy \tag{27}$$

The conditions on φ which have been imposed in (14) are not sufficient to imply that this integral converges to zero as $R \to \infty$. We shall presently have to impose several conditions upon G and at that time will add the necessary requirements to ensure vanishing of the above integral as $R \to \infty$. In the meantime we shall assume this, and also the vanishing of that part of the contour integral in (20) taken around the large cylinder.

We now have the following formula:

$$\varphi(x, y, z) = (23) + (20), \tag{28}$$

where the two line integrals around the ship combine and reduce to the form (26).

If this expression is made dimensionless by measuring lengths by $\ell = L/2$ and velocities by c, so that

$$\varphi(x,y,z) = c\,\ell\tilde{\varphi}\left(\frac{x}{\ell}, \frac{y}{\ell}, \frac{z}{\ell}\right) = c\,\ell\tilde{\varphi}(\tilde{x},\tilde{y},\tilde{z}), \quad G = \frac{1}{\ell}\tilde{G}, \tag{29}$$

we obtain, <u>after dropping the tildes</u>,

$$\begin{aligned}
\varphi(x,y,z) = &-\frac{1}{4\pi}\iint_{S_{FP}} [c^2(g\ell)^{-1}G_{\xi\xi}(P;\xi,Y,\zeta) + G_\eta]\varphi(\xi,Y,\zeta)\,d\xi d\zeta \\
&+\frac{1}{4\pi}\iint_{S_{FP}} \{-c^2(g\ell)^{-1}[\varphi G_{\xi\eta} + \varphi_\eta G_\xi]Y_\xi + G_\eta YY_\xi \\
&+[Y_\xi G_\xi + Y_\zeta G_\zeta]\varphi - \frac{1}{2}c^2(g\ell)^{-1}[\varphi_\xi^2 + \varphi_\eta^2 + \varphi_\zeta^2]G_\xi\}\,d\xi d\zeta \\
&-\frac{1}{4\pi}\iint_{S_{wP}} f(\xi',\eta')[G_\xi(P;\xi,\eta,f) + G_\xi(P;\xi,\eta,-f)]\,d\xi' d\eta' \\
&-\frac{1}{4\pi}\iint_{S_{wP}} [\varphi(\xi,\eta,f)\{[f_{\xi'}\cos\alpha - f_{\eta'}\sin\alpha][G_\xi^+ + G_\xi^-] \\
&+[f_{\xi'}\sin\alpha + f_{\eta'}\cos\alpha][G_\eta^+ + G_\eta^-] - [G_\zeta^+ - G_\zeta^-]\} \\
&+f(\xi',\eta')[f_{\xi'}\cos\alpha - f_{\eta'}\sin\alpha][G_\zeta^+ - G_\zeta^-]]\,d\xi' dy' \\
&+\frac{1}{4\pi}\int_{L_P} [G(P;\xi,\hat{Y},\hat{f}) + G(P;\xi,\hat{Y},-\hat{f})][\hat{f}\hat{Y}' - \hat{f}\hat{Y}]\,d\xi \\
&+\frac{1}{4\pi}\int_{L_P} c^2(g\ell)^{-1}[G_\xi^+ + G_\xi^-]\varphi(\xi,\hat{Y},f)\,\hat{f}'d\xi \,.
\end{aligned} \tag{30}$$

We are now ready to define more precisely the form of the Green's function $G(x,y,z;\xi,\eta,\zeta)$, where the variables are the dimensionless ones used in (30). We shall suppose that, in addition to (15) and (17), G satisfies the following conditions

$$c^2(g\ell)^{-1}\, G_{\xi\xi}(x,y,z;\xi,0,\zeta) + G_\eta(x,y,z;\xi,0,\zeta) = 0,$$

$$G(x,y,z;\xi,\eta,\zeta) = \begin{cases} O([\xi^2+\zeta^2]^{-1/2}) & \text{as } \xi^2+\zeta^2 \to \infty \text{ for } \xi < 0, \\ O(1) & \text{as } \xi^2+\zeta^2 \to \infty \text{ for } \xi > 0. \end{cases}$$

(30)

The solution for G is well known and may be expressed as follows:

$$G(x,y,z;\xi,\eta,\zeta) = [(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2]^{-1/2} + \zeta(x-\xi)^2$$

$$+ (y+\eta+2h)^2 + (z-\zeta)^2]^{-1/2}$$

$$- \frac{4}{\pi}\, \gamma_0 \int_0^{1/2\pi} d\theta \oint_0^\infty \frac{e^{-k\gamma_0 h}\operatorname{sech} k\gamma_0 h \cosh k\gamma_0(\eta+h)[\cosh k\gamma_0(y+h)(k\cos^2\theta + 1)-1]}{k\cos^2\theta - \tanh k\gamma_0 h}\, x$$

$$\cos[k\gamma_0(x-\xi)\cos\theta]\cos[k\gamma_0(z-\zeta)\sin\theta]\, dk$$

$$- 4\gamma_0 \int_{\theta_0}^{1/2\pi} \frac{e^{-k_0\gamma_0 h}\operatorname{sech} k_0\gamma_0 h \cosh k_0\gamma_0(\eta+h)[\cosh k_0\gamma_0(y+h)(k_0\cos^2\theta+1)-1]}{\cos^2\theta - \gamma_0 h \operatorname{sech}^2 k_0\gamma_0 h}\, x$$

$$\sin[k_0\gamma_0(x-\xi)\cos\theta]\cos[k_0\gamma_0(z-\zeta)\sin\theta]\, d\theta ,$$

(32)

where

$$\gamma_0 = \frac{g\ell}{c^2}, \quad \theta_0 = \begin{cases} \arccos\sqrt{\gamma_0 h} & \text{if } \gamma_0 h \le 1 \\ 0 & \text{if } \gamma_0 h \ge 1 \end{cases}$$

and where $k_0 = k_0(\theta)$ is the real positive root of

$$k_0 - \sec^2\theta \tanh k_0\gamma_0 h = 0, \quad \theta_0 < \theta < \tfrac{1}{2}\pi .$$

The integral with respect to k in (32) must be interpreted as a Cauchy principal value if $\theta_o < \theta < 1/2\pi$. The combination $\gamma_o h$ occurring throughout is just the depth Froude number, i.e., gh/c^2 in the dimensional variables. If the depth of fluid is infinite, one may obtain either directly or from (32) by letting $\gamma_o h \to \infty$ the following simpler expression:

$$G(x,y,z;\xi,\eta,\ ) = [x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2]^{-1/2} - [(x-\xi)^2 + (y+\eta)^2 + (z-\zeta)^2]^{-1/2}$$

$$- \frac{4}{\pi}\gamma_o \int_0^{1/2\pi} d\theta \oint_0^{\infty} \frac{e^{k\gamma_o(y+\eta)}}{k\cos^2\theta - 1} \cos[k\gamma_o(x-\xi)\cos\theta]\cos[k\gamma_o(z-\zeta)\sin\theta]dk$$

$$- 4\gamma_o \int_0^{1/2\pi} e^{\gamma_o(y+\eta)\sec^2\theta} \sin[\gamma_o(x-\xi)\sec\theta]\cos[\gamma_o(z-\zeta)\sin\theta\sec^2\theta]\sec^2\theta d\theta, \tag{33}$$

where $\gamma_o = g\ell/c^2$ as in (32).

We note in passing that the integral terms in both (32) and (33) are functions of $\gamma_o(x-\xi)$, $\gamma_o(y+\eta)$, $\gamma_o(z-\zeta)$, so that the independence upon x, y, z, ξ, η, ζ and γ_o is not quite as complicated as it appears. For (33) it is, in fact, possible to express the integrals in terms of the exponential integral. Let H(x,y,z) be the function obtained when $\gamma_o(x-\xi)$, $\gamma_o(y+\eta)$, $\gamma_o(z-\zeta)$ are replaced by x,y,z in the integral terms. Then

$$H(x,y,z) = -\frac{4}{\pi}\gamma_o \int_0^{2\pi} \exp\{[y+i(x\cos\theta + z\sin\theta)]\sec^2\theta\} \times \mathrm{Ei}\{-[y+i(x\cos\theta + z\sin\theta]\sec^2\theta\}\sec^2\theta d\theta \tag{34}$$

The function G for either (32) or (33) has now a property which restricts the possible positions of the field point (x,y,z). There is a singularity not only at (x,y,z) but also at (x,-y,z) (the one at (x,-y-2h,z) does not bother us). Hence, if Y(x,z) > 0 and if the point (x,y,z) is chosen such that -Y < y < Y the function $G - r^{-1}$ will not be harmonic within the fluid. Unless we assume

$$-h < y < -|Y(x,z)| , \tag{35}$$

we cannot use (16) together with (32). However, with this proviso the function G is defined and has the desired behavior not only for (ξ, η, ζ) in the fluid, but also for points such that $\eta \leq |Y|$ and for points lying within the volume bounded by the hull and its reflection in the plane $y = 0$.

It is now possible to verify that (28) is satisfied. An investigation of the asymptotic behavior of G, G_ξ and G_R shows that all are $O([\xi^2 + \zeta^2]^{-1})$ as $\xi^2 + \zeta^2 \to \infty$ for $\xi < 0$ and $o([\xi^2 + \zeta^2]^{-1/8})$ for $\xi > 0$. From this and (14) it follows easily that (28) and the line integral along the intersection of S_R and S_F occurring in (20) both vanish as $R \to \infty$.

In the dimensionless representation used in (30) the ordinate at $(\xi', \eta') = (0,0)$ is $/\ell = B/L$. Let us denote this parameter B/L by ϵ and display it by introducing a hull $f^{(1)}(\xi', \eta')$ with $f^{(1)}(0,0) = 1$ which is related affinely to $f(\xi', \eta')$ by

$$f(\xi', \eta') = \epsilon f^{(1)}(\xi', \eta') \quad . \tag{36}$$

It is evident that the function φ, the free surface Y, the quantities α and e, and the force and moment will also depend upon the parameter ϵ:

$$\varphi(x, y, z; \epsilon), \quad Y(x, z; \epsilon), \quad \alpha(\epsilon), \quad e(\epsilon), \quad \text{etc.} \tag{37}$$

If one now examines the integrals in (30) involving f, they are all clearly $O(\epsilon)$ except possibly the one involving $G_\zeta^+ - G_\zeta^-$. It is not difficult to see that this term is also $O(\epsilon)$. For example, if one starts with (33), one may easily verify that

$$G_\zeta(P; \xi, \eta, f) - G_\zeta(P; \xi, \eta, -f)$$

$$= f \cdot \left\{ \frac{1}{r_+^3 r_-^3} \left[4z^2 \frac{r_+^2 + r_+ r_- + r_-^2}{r_+ + r_-} - (r_+^3 + r_-^3) \right] - \right.$$

$$\left. - \frac{1}{r_{1+}^3 r_{1-}^3} \left[4z^2 \frac{r_{1+}^2 + r_{1+} r_{1-} + r_{1-}^2}{r_{1+} + r_{1-}} - (r_{1+}^3 + r_{1-}^3) \right] \right\}$$

$$+ \frac{8}{\pi} \gamma_0^2 \int_0^{1/2\pi} d\theta \oint_0^\infty \frac{k e^{k\gamma_0(y+\eta)}}{k \cos^2\theta - 1} \cos[k\gamma_0(x-\xi)\sec\theta]$$

$$\cos[k\gamma_0 z \sin\theta] \sin[k\gamma_0 f \sin\theta] \, dk$$

$$+ 8\gamma_o^2 \int_0^{1/2\pi} e^{\gamma_o(y+\eta)\sec^2\theta} \sin[\gamma_o(x-\xi)\sec\theta] \cos[\gamma_o z \sin\theta\sec^2\theta]\sin[\gamma_o f\sin\theta\sec^2\theta]\, d\theta, \tag{38}$$

where

$$r_+^2 = (x-\xi)^2 + (y-\eta)^2 + (z+f)^2, \qquad r_{1+}^2 = (x-\xi)^2 + (y+\eta)^2 + (z+f)^2,$$

$$r_-^2 = (x-\xi)^2 + (y-\eta)^2 + (z-f)^2, \qquad r_{1-}^2 = (x-\xi)^2 + (y+\eta)^2 + (z-f)^2.$$

But then φ itself must be $O(\epsilon)$, an estimate consistent with the remaining integrals since, if φ is $O(\epsilon)$, then also Y is $O(\epsilon)$ from (6). One notices next that, if φ and Y are $O(\epsilon)$, then most of the integrals in (30) are $O(\epsilon^2)$. In fact, the only ones not evidently so are the first and third. However, as long as (35) is satisfied, the integrand in the first integral may be expanded in a series in Y. Since the first term will vanish because of the boundary condition (31), this integral is also $O(\epsilon^2)$. There remains the third integral. Although this can now stand as it is, it may also be simplified by decomposing it into a part which is $O(\epsilon)$ and a remainder which is $O(\epsilon^2)$. If we admit for a moment that α and e are each $O(\epsilon)$ and then make use of (2), we find

$$G_\xi(P;\xi,\eta,f) + G_\xi(P;\xi,\eta,-f) = 2G_\xi(P;\xi',\eta',0) + O(\epsilon^2). \tag{39}$$

Furthermore, the area which is the symmetric difference* between S_{WP} and S_O, the value of S_{WP} when the ship is at rest, is also $O(\epsilon)$, so that the error is $O(\epsilon^2)$ if one replaces the integral over S_{WP} by one over S_O. On finds then finally the classical thin-ship result

$$\varphi(x,y,z) = -\frac{1}{2\pi} \iint_{S_O} f(\xi',\eta')G_\xi(x,y,z;\xi',\eta',0)d\xi' d\eta' + O(\epsilon^2). \tag{40}$$

Still other information can be obtained from (30). For example, it is sometimes asserted that the approximation (40) requires smallness not only of f (i.e. f/L if dimensions are reintroduced) but also of $f_{\xi'}$ and $f_{\eta'}$. As long as f is bounded, which it must be from physical considerations, steep slopes cannot cause any of the $O(\epsilon^2)$ integrals in (30) involving $f_{\xi'}$ or $f_{\eta'}$ to become large. Consider an integral of the

*In the notation of set theory this is $S_{WP} + S_O - S_{WP} \cdot S_O$.

$$\iint A(\xi',\eta',f)f_{\eta'}\, d\xi' d\eta'$$

If, for a fixed value of ξ', $f(\xi',\eta')$ is an increasing function of η' from keel to intersection with the free surface, one may write $\eta' = \eta'(f)$ and express the integral in the form

$$\int d\xi' \int_{0}^{\hat{f}(\xi')} A(\xi',\eta'(f),\ f)df,$$

which is evidently $O(\epsilon)$ without any assumption about smallness of $f_{\eta'}$. If the section form is something like those shown in Figure 5, one must divide the region of integration into appropriate parts and treat

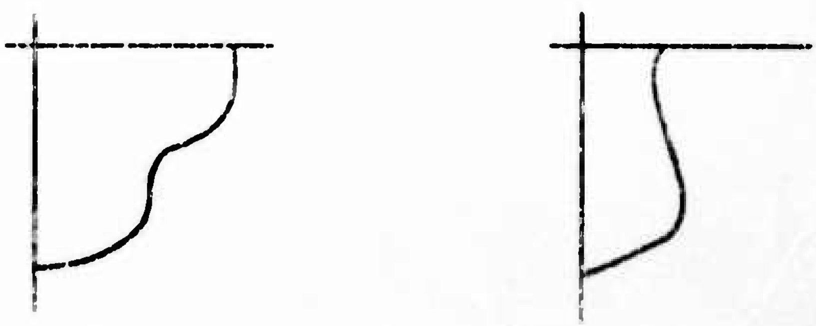

Figure 5

each separately. The result is the same. A similar result holds for the integrals involving $f_{\xi'}$. However, the approximation (40) may break down in the neighborhood of the under-water profile, where singularities may occur in the expression (40) unless f_x is Hölder continues there, even though they are not properly present in φ itself. To require Hölder continuity at the stem and stern would be an intolerable restriction if the theory is to be applicable to real ships. One could try to improve the approximation here by the method used by Lighthill for the thin-wing approximation. However, if one is chiefly interested in the resistance, which can be determined from the behavior of φ far from the ship, this is not necessary. On the other hand, if one is concerned with finding ships of minimum wave resistance, one should properly try to take this into account.

There is another point with respect to (40) which has been made by Maruo (1962). Consider, for example, (38). In the integral terms f always has γ_0 as a factor. The same statement applies to some of the $O(\epsilon^2)$ terms in (39). Since γ_0 varies from about 2 to 12

for the speed range of most ships, it appears that for a given form $f^{(1)}$, the value of ϵ must be smaller for lower speeds than for higher ones in order for (40) to give the same degree of approximation. However, the conclusion is not certain, for increasing γ_0 also decreases the values of the integrals. What is needed is more precise knowledge of the behavior of the function G. Furthermore, inspection of the first, second and last integrals in (30) shows that $1/\gamma_0$ occurs as a factor in various terms. One might reasonably assume that a small value of γ_0 would magnify the effect of these terms, thus acting now in the opposite direction. The physical argument would appear to be that large Froude numbers make the linearized free-surface condition less exact, whereas small Froude numbers make the thin-body approximation less exact.

Nothing has been said as yet concerning the ratio B/H which for normal displacement ships is likely to be between 2 and 4. Let us introduce the parameter $\delta = H/L$ and suppose that varying δ results in affine variations of the ship form. Then the functions in (37) depend upon δ as well as ϵ. In (30) δ enters explicitly only through the centerplane region S_{wP} whose area is evidently $O(\delta)$. The third integral in (30) is now $O(\epsilon\delta)$, so that φ itself, and also Y, must be $O(\epsilon\delta)$. But then it is easy to verify that the first, second, and fourth integrals are $O(\epsilon^2\delta^2)$ or $O(\epsilon^3\delta)$ (for the last term of the fourth integral) and that the last two line integrals are $O(\epsilon^2\delta)$. Furthermore, the difference between the third integral and (40) is $O(\epsilon^2\delta)$. This difference can also be written as a line integral along L_P with neglect of only higher-order terms. From these formal considerations it appears that, at least at some distance from the ship, the expression in (40) will still include the most important part of φ when both ϵ and δ are small. Furthermore, the next most important contribution would appear to be the line integrals along the load waterline. If one uses (40) to compute φ and Y where these occur in the line integrals, they may be expressed as follows, with errors being of higher order:

$$-\frac{1}{2\pi}\int_{L_P} G(P;\xi',0,0)[f(\xi',0)\varphi_\eta(\xi',0,0) + f_{\xi'}(\xi',0)\varphi_\xi(\xi',0,0)]d\xi'$$

$$+\frac{1}{2\pi}\frac{1}{\gamma_0}\int_{L_P} G_\xi(P;\xi',0,0)[f_{\xi'}(\xi',0)\varphi(\xi',0,0) - f(\xi',0)\varphi_\xi(\xi',0,0)]d\xi' \tag{41}$$

This improvement would, of course, have little relation to a flat-ship theory where no assumption is made about ϵ.

The expression (30) may also be used to derive in a convenient manner successive terms in a perturbation expansion for the thin ship. As is usual in this procedure, we assume the expansions

$$
\begin{aligned}
\varphi(x,y,z;\epsilon) &= \epsilon\,\varphi^{(1)}(x,y,z) + \epsilon^2\varphi^{(2)} + \ldots\ldots, \\
Y(x,z;\epsilon) &= \epsilon\,Y^{(1)}(x,z) + \epsilon^2 Y^{(2)} + \ldots\ldots, \\
\alpha(\epsilon) &= \epsilon\,\alpha^{(1)} + \epsilon^2\alpha^{(2)} + \ldots\ldots, \\
e(\epsilon) &= \epsilon\,e^{(1)} + \epsilon^2 e^{(2)} + \ldots\ldots,
\end{aligned}
\qquad (42)
$$

etc. Substitution in (30), expansion in Taylor's series, and rearrangement according to powers of ϵ yield the following expressions for $\varphi^{(1)}$ and $\varphi^{(2)}$:

$$
\varphi^{(1)}(x,y,z) = -\frac{1}{2\pi}\iint_{S_0} f^{(1)}(\xi',\eta')G_\xi(x,y,z;\xi',\eta',0)\,d\xi' d\eta' ,
$$

$$
\begin{aligned}
\varphi^{(2)}(x,y,z) = &-\frac{1}{4\pi}\int_{-\infty}^{\infty}\!\!\int \left[\frac{1}{\gamma_0} G_{\xi\xi\eta}(P;\xi,0,\zeta) + G_{\eta y}\right]Y^{(1)}(\xi,\zeta)\varphi^{(1)}(\xi,0,\zeta)\,d\xi d\zeta \\
&+\frac{1}{4\pi}\int_{-\infty}^{\infty}\!\!\int \Big\{-\frac{1}{\gamma_0}\left[\varphi^{(1)}G_{\xi\eta} + \varphi_\eta^{(1)}G_\xi\right]Y_\xi^{(1)} + G_\eta Y^{(1)}Y_\xi^{(1)} \\
&+\left[Y_\xi^{(1)}G_\xi + Y_\zeta^{(1)}G_\zeta\right]\varphi^{(1)} - \frac{1}{2\gamma_0}\left|\operatorname{grad}\varphi^{(1)}\right|^2 G_\xi\Big\}\,d\xi d\zeta \\
&+\frac{1}{2\pi}\,\alpha^{(1)}\iint_{S_0}\left[-\eta G_{\xi\xi}(P;\xi,\eta,0) + \xi G_{\xi\eta}\right] f(\xi,\eta)\,d\xi d\eta \\
&+\frac{1}{2\pi}\,e^{(1)}\iint_{S_0} G_{\xi\eta} f\,d\xi d\eta \qquad (43)\\
&-\frac{1}{2\pi}\iint_{S_0}\varphi^{(1)}(\xi,\eta,0)\left\{f_\xi^{(1)}G_\xi + f_\eta^{(1)}G_\eta - f^{(1)}G_{\zeta\zeta}\right\}d\xi d\eta \\
&-\frac{1}{2\pi}\int_{L_p} G_\xi(P;\xi,0,0)\,f^{(1)}(\xi,0)\,Y^{(1)}(\xi,0)\,d\xi \\
&+\frac{1}{2\pi}\int_{L_p}\left\{G\left[f^{(1)}Y_\xi^{(1)} - f_\xi^{(1)}Y^{(1)}\right] + \frac{1}{\gamma_0}G_\xi\varphi^{(1)}(\xi,0,0)f_\xi^{(1)}\right\}d\xi
\end{aligned}
$$

$\varphi^{(1)}$ is, of course, just the Michell potential (40); $\varphi^{(2)}$ is usually not written down, and the reason seems obvious. However, since it has been,

a few comments are necessary. The function φ itself is defined only where there is water. However, the expression for $\varphi^{(1)}$ is defined for the whole lower half-space, and hence $Y^{(1)}$ may be defined for the whole (x, z)-plane. This has permitted some simplification in the integrals occuring in the expression for $\varphi^{(2)}$ without changing its order. In particular, (x, z)-plane, and evaluating many terms in the integrand on either the plane $\eta = 0$ or $\zeta = 0$ or their intersection.

A point of interest in $\varphi^{(2)}$ is that the second-order corrections to the third integral in (30) bring in the trim and sinkage and the wave profile (the first line integral) but not the fact that dipoles on the surface are replaced by dipoles on the centerplane. The correction for this latter approximation enters only with ϵ^3.

The potential $\varphi^{(2)}$ is not yet completely determined because $\alpha^{(1)}$ and $e^{(1)}$ have not been determined. In order to find these we must turn to the dynamic boundary conditions (11).

In order to compute the force and moment components according to the formulas in (9), one needs to know p, which can be found from (4) with φ_t deleted and $p_a = 0$. If we make use of (42) and compute p from a formal expansion, we find

$$p = -\rho g y + \rho c \varphi_x + O(\epsilon^2) \quad ,$$

or, in terms of the dimensionless variables used in (30)

$$p = -\rho g \ell y + \rho c^2 \varphi_x + O(\epsilon^2). \tag{44}$$

We shall not dwell upon the details of substitution in (9) and computation of the integrals correct to formal order $O(\epsilon^3)$, for this is straight-forward if tedious, and in any case well known. (Use of the integration-by-parts formula (18) simplifies somewhat the handling of the hydrostatic term in (44).) Equations (11) constrain one to assume that $T = O(\epsilon^2)$. One then finds from (11) the following relations among first-order quantities, the variables in the integrals all being the dimensionless ones in (30):

$$\frac{T}{2\rho g \ell^3} = -\frac{1}{\gamma_o} \iint_{S_o} \varphi_x(x,y,0)\, f_x(x,y)\, dxdy + O(\epsilon^3), \int_{L_P} (e+\alpha x) f(x,0) dx$$

$$-\frac{1}{\gamma_o} \iint_{S_o} \varphi_x(x,y,0)\, f_x(x,y)\, dxdy = O(\epsilon^3),$$

$$\frac{T}{2\rho g \ell^3}\frac{d}{\ell} - \int_{L_P} (e+\alpha x)(x-x_G) f(x,0) dx - \alpha \iint_{S_o} (y-y_G) f(x,y)\, dxdy$$

$$-\frac{1}{\gamma_o} \iint_{S_o} [(x-x_G)f_y - (y-y_G)f_x]\varphi_x(x,y,0)\, dxdy = O(\epsilon^3). \tag{45}$$

In (44) and (45) φ is, of course, to be taken at the expression (40). However, there is a difficulty. For the validity of (40) we needed essentially only the smallness of the beam/length ratio as long as the point (x,y,z) is not near the ship. For points on the hull further conditions may be necessary for (40) to be a good approximation. The difficulty mentioned above is that if we use (40) to compute (45) we will be assuming that it is accurate also on the hull. This seems to be unavoidable in finding $\alpha^{(1)}$ and $e^{(1)}$ and hence $\varphi^{(2)}$ in (43). However, this is not so for the resistance, i.e., the first equation in (45). It is known, at least since a paper of Havelock's (1932), that one can compute the resistance by either momentum- or energy-flux considerations at a great distance from the ship. This leads just to the Michell integral or to Sretenskii's generalizations for finite depth or canals of finite width, but in the present case modified so that the formulas are expressed in terms of f rather than f_x. For example, for finite depth

$$\frac{T}{2\rho g \ell^3} = \frac{\gamma_o}{\pi} \iint_{S_o} dxdy \iint_{S_o} d\xi d\eta\, f(x,y) f(\xi,\eta) \int_{\mu_h}^{\infty} \frac{\cosh \mu(y+h) \cosh \mu(y+h)}{\cosh^2 \mu h} \times$$

$$\cos[(x-\xi)(\gamma_o \mu \tanh \mu h)^{1/2}]\mu \tanh \mu h \left[\frac{\mu}{\mu - \gamma_o \tanh \mu h}\right]^{1/2} d\mu\,, \tag{46}$$

where μ_h is the positive solution of $\mu_h = \gamma_o \tanh \mu_h h$, or 0 if this doesn't exist.

What is important here is not the specific formula but the conclusion that the resistance as computed by thin-ship theory is the most important term in the "exact" wave resistance as long as the beam/length is small, without further restrictions upon the beam/draft ratio or the slopes.

Unsteady Motion

A procedure similar to that for steady motion may now be carried through for unsteady motion. It will be convenient to assume that the ship starts from rest in still water at time $t = 0$. Instead of applying Green's formula to φ as in (16) we shall now apply it to $\phi_t(\bar{x},\bar{y},\bar{z},t)$:

$$\phi_t(\bar{x},\bar{y},\bar{z},t) = \frac{1}{4\pi} \iint [\phi_{t\nu}(Q)G(P;Q) - \phi_t(Q)G_\nu(P;Q)]\, dS(Q) \qquad (47)$$

Here and in the following all variables have their proper dimensions. We can immediately dispose of the integrals over S_B and S_R by requiring that G satisfy

$$G_\eta(P;\xi,-h,\zeta) = 0, \qquad G = O([\xi^2 + \zeta^2]^{-1}) \text{ as } \xi^2 + \zeta^2 \to \infty \qquad (48)$$

or a statement (see (17')) concerning the limit of G_η if the depth is infinite.

For the integral over S_F we note that from (6)

$$\phi_t = -g\,\bar{Y} - \frac{1}{2}\,|\text{grad}\,\phi|^2 ,$$

$$\phi_\nu = \bar{Y}_t/[1 + \bar{Y}^2_{\bar{x}} + \bar{Y}^2_{\bar{z}}]^{1/2} , \qquad (49)$$

$$\phi_{\nu t} = \frac{\bar{Y}_{tt}}{[1 + \bar{Y}^2_{\bar{x}} + Y^2_{\bar{z}}]^{1/2}} - \frac{\bar{Y}_t(\bar{Y}_{\bar{x}}\bar{Y}_{\bar{x}t} + \bar{Y}_{\bar{z}}\bar{Y}_{\bar{z}t})}{[1 + \bar{Y}^2_{\bar{x}} + \bar{Y}^2_{\bar{z}}]^{3/2}} - \phi_{\nu y}Y_t$$

and that

$$G_\nu(P;Q) = [G_{\bar{\eta}} - G_{\bar{\xi}}\bar{Y}_{\bar{\xi}} - G_{\bar{\zeta}}\bar{Y}_{\bar{\zeta}}][1 + \bar{Y}^2_{\xi} + \bar{Y}^2_{\zeta}]^{-1/2},$$

so that

$$\frac{1}{2\pi}\iint_{S_F}[\phi_{t\nu}G - \phi_t G_\nu]\,dS$$
$$= \frac{1}{4\pi}\iint_{S_{FP}}[\bar{Y}_{tt}(\bar{\xi},\bar{\zeta},t)G(P;\bar{\xi},\bar{Y},\bar{\zeta}) + g\bar{Y}G_\eta]d\bar{\xi}d\bar{\zeta} + O(\epsilon^2) \tag{50}$$

Here we have anticipated that $\bar{Y}$ and ϕ will each be $O(\epsilon)$.

We now subject G to further conditions. We shall suppose that G is a symmetric function of t satisfying the following boundary and initial conditions:

$$\begin{aligned} &G_{tt}(\bar{x},\bar{y},\bar{z};\bar{\xi},0,\bar{\zeta};t) + g\,G_{\bar{\eta}} = 0, \\ &G(\bar{x},\bar{y},\bar{z};\bar{\xi},0,\bar{\zeta};0) = G_t(\bar{x},\bar{y},\bar{z};\bar{\xi},0,\bar{\zeta};0) = 0. \end{aligned} \tag{51}$$

Methods of construction of such Green's functions are well known (see, e.g., Wehausen and Laitone, 1960, pp. 491-495). For finite depth G may be expressed as follows:

$$\begin{aligned} G(x,y,z;\xi,\eta,\zeta;t) &= \frac{1}{r} + \frac{1}{r_2} \\ &- 2\int_0^\infty e^{-kh}\,\frac{\cosh k(\bar{y}+h)\cosh k(\bar{\eta}+h)}{\cosh kh}\,J_0(kR)\,dk \\ &+ 4\int_0^\infty [1-\cos\sigma(k)t]\,\frac{\cosh k(\bar{y}+h)\cosh k(\bar{\eta}+h)}{\sinh 2kh}\,J_0(kR)\,dk, \end{aligned} \tag{52}$$

where $\sigma(k) = [gk\tanh kh]^{1/2}$, and for infinite depth as

$$\begin{aligned} G(\bar{x},\bar{y},\bar{z};\bar{\xi},\bar{\eta},\bar{\zeta},t) &= \frac{1}{r} - \frac{1}{r_1} \\ &+ 2\int_0^\infty e^{k(\bar{y}+\bar{\eta})}\,J_0(kR)\,[1-\cos\sqrt{gk} +]\,dk = \frac{1}{r} + \frac{1}{r_1} \\ &- 2\int_0^\infty e^{k(\bar{y}+\bar{\eta})}\,J_0(kR)\cos\sqrt{gkt}\,dk, \end{aligned} \tag{53}$$

where

$$r^2 = (\bar{x}-\bar{\xi})^2 + (\bar{y}-\bar{\eta})^2 + (\bar{z}-\bar{\zeta})^2, \quad r_1^2 = (\bar{x}-\bar{\xi})^2 + (\bar{y}+\bar{\eta})^2 + (\bar{z}-\bar{\zeta})^2 ,$$

$$r_2^2 = (\bar{x}-\bar{\xi})^2 + (\bar{y}+\bar{\eta}+2h)^2 + (\bar{z}-\bar{\zeta})^2; \; R^2 = (\bar{x}-\bar{\xi})^2 + (\bar{z}-\bar{\zeta})^2 .$$

In both formulas $G_t(\bar{x},\bar{y},\bar{z};\bar{\xi},\bar{\eta},\bar{\zeta};0) = 0$ although (50) required this only for $\bar{\eta} = 0$.

Under condition (35) we may now use (51) to replace the term $gG_{\bar{\eta}}$ in (50) by $-G_{tt}$. However, there is nothing to require us to use $G(P;Q;t)$ for this purpose. Instead, for reasons which will become apparent later, we shall replace t by τ in (51) and in G use the argument $t-\tau$. Then (50) becomes

$$\frac{1}{4\pi} \iint_{S_{FP}(\tau)} \bar{Y}_{\tau\tau}(\bar{\xi},\bar{\zeta},\tau)G(P;\bar{\xi},0,\bar{\zeta};t-\tau) - \bar{Y}G_{tt} \; d\bar{\xi}d\bar{\zeta} + O(\epsilon^2)$$

Consider now the following integral:

$$\frac{d}{d\tau}\frac{1}{4\pi} \iint_{S_{FP}(\tau)} [\bar{Y}_\tau(\bar{\xi},\bar{\zeta},\tau)G(P;\bar{\xi},0,\bar{\zeta};t-\tau) + \bar{Y}G]\; d\bar{\xi}d\bar{\zeta}$$

$$= \frac{1}{4\pi} \iint_{S_{FP}} [\bar{Y}_{\tau\tau}G - \bar{Y}_\tau G_t + \bar{Y}_\tau G_t - \bar{Y}G_{tt}]\; d\bar{\xi}d\bar{\zeta}$$

$$+ \frac{1}{4\pi} \oint_{C_{FP}(\tau)} [\bar{Y}_\tau G + \bar{Y}G_t]\; U_n ds \quad , \tag{54}$$

where $C_{FP}(\tau)$ is the contour bounding (internally) S_{FP} and U_n is the velocity at each point of the contour in the direction of its (internal) normal. By a modification of the computation used further below to find V_n one can show that

$$U_n = \dot{\bar{x}}_o f_{x'}(x',0) + O(\epsilon) ,$$

so that the contour integral is $O(\epsilon^2)$ since f is $O(\epsilon)$. Hence (51) may be replaced by

$$\frac{d}{d\tau}\frac{1}{4\pi} \iint_{S_{FP}(\tau)} [\bar{Y}_\tau G + \bar{Y}G_t]\; d\bar{\xi}d\bar{\zeta} + O(\epsilon^2) \tag{55}$$

Let us turn now to the integral over S_w. Here we have, of course, the same difficulty as before, namely, that it is easier to express this integral in the coordinates (ξ', η', ζ'). Fortunately, we can borrow from the steady-state computations. In particular, the terms associated with the integral of $\phi_t G_\nu$ will be the same as the last three terms in (22) with φ replaced by ϕ_t. Evidently these terms will be $O(\epsilon^2)$.

Consider now

$$\frac{1}{4\pi} \iint_{S_w} \phi_{\nu t} G \, dS \; . \tag{56}$$

If we express this in ship coordinates we can write it as follows:

$$\begin{aligned} \frac{1}{2\pi} \iint_{S_{wP}(\tau)} & \phi_{\nu t}(\bar{x}_o(\tau) + \xi'\cos\alpha(\tau) - \eta'\sin\alpha, e(\tau) + \xi'\sin\alpha + \\ & + \eta'\cos\alpha, \; f(\xi', \eta'), \tau) \times G(\bar{x}, \bar{y}, \bar{z}; \bar{x}_o + \ldots, f; t-\tau) \\ & [1 + f^2_{\xi'} + f^2_{\eta'}]^{1/2} \, d\xi' d\eta' \end{aligned} \tag{57}$$

where x_o, α and e are functions of τ. We have again replaced t by τ and taken the last variable in G as $t-\tau$. We now have the following identity:

$$\begin{aligned} \frac{d}{d\tau} \frac{1}{2\pi} \iint_{S_{wP}(\tau)} \phi_\nu G \, [1 + f^2_{\xi'} + f^2_{\eta'}]^{1/2} d\xi' d\eta' &= \frac{1}{2\pi} \iint \phi_{\nu t} G[\ldots]^{1/2} d\xi' d\eta' \\ + \frac{1}{2\pi} \iint_{S_{wP}} \{\phi_{\nu\xi}[\dot{\bar{x}}_o - (\xi'\sin\alpha + \eta'\cos')\alpha'] &+ \phi_{\nu\eta}[\dot{e} + (\xi'\cos\alpha - \eta'\sin\alpha)\alpha']\} \\ & \times G \cdot [\ldots]^{1/2} d\xi' d\eta' \\ + \frac{1}{2\pi} \iint_{S_{wP}} \phi_\nu [\ldots]^{1/2} \{-G_t + G_\xi[\dot{\bar{x}}_o - (\ldots)\dot{\alpha}] &+ G_\eta[\dot{e} + (\ldots)\dot{\alpha}]\} d\xi' d\eta' \\ + \frac{1}{2\pi} \oint_{C_{wP}} \phi_\nu \, [\ldots]^{1/2} \, G \, U_n ds \quad , \end{aligned} \tag{58}$$

where C_{wP} is the contour bounding S_{wP} and U_n is the velocity at each point of the contour in the direction of the exterior normal to S_{wP} at the point. Since the integrations are being carried out in ship coordinates, the only part of C_{wP} where U_n is not zero is along the

projection of the waterline. Here

$$U_n = - \frac{[x'\cos\alpha - y'\sin\alpha]\alpha - \bar{Y}_{\bar{x}}[\dot{\bar{x}}_o - (x'\sin\alpha + y'\cos\alpha)\alpha] - \bar{Y}_t}{[1 + Y_{\bar{x}}^2 + Y_{\bar{z}}^2]^{1/2}}$$

$$= \dot{\bar{x}}_o \bar{Y} + \bar{Y} + O(\epsilon^2) = O(\epsilon). \qquad (59)$$

Evidently this integral will be $O(\epsilon^2)$. (We note in passing that the contour integral would have made an important contribution if we had carried out the integrations in the $\bar{O}\ \bar{x}\ \bar{y}\ \bar{z}$ coordinates; this is now contained in the two preceding integrals.) If we anticipate not that α, $\dot{\alpha}$, e and $\dot{e}$ are $O(\epsilon)$ along with f and ϕ, it is evident that several terms in (58) are $O(\epsilon^2)$ and that we may write

$$\frac{1}{2\pi} \int\int_{S_{wP}(\tau)} \phi_{\nu t}\, G\, [\ldots]^{1/2}\, d\xi' d\eta'$$

$$= \frac{d}{d\tau} \frac{1}{2\pi} \int\int_{S_{wP}(\tau)} \phi_\nu [\ldots] \quad G\, d\xi' d\eta' + \frac{1}{2\pi} \int\int_{S_{wP}(\tau)} \phi_\nu [\ldots]^{1/2}\, G_t d\xi' d\eta'$$

$$- \frac{1}{2\pi} \int\int_{S_{wP}(\tau)} \dot{\bar{x}}_o(r)[\phi_{\nu\xi} G + \phi_\nu G_\xi][\ldots]^{1/2}\, d\xi' d\eta' + O(\epsilon^2).$$

These integrals can be simplified still more without changing the order of the remainder. In particular, one can replace $S_{wP}(\tau)$ by $S_{wP}(o) = S_o$ and take the argument of G so that it appears as

$$G(x,y,z; x(\tau) + \xi', \eta', 0; t-\tau) .$$

Let use next compute ϕ_ν which according to the kinematic boundary condition must equal V_ν, the component of the hull velocity in the direction of the interior normal. Here

$$\vec{V} = (\dot{\bar{x}}_o - [x'\sin\alpha + y'\cos\alpha]\dot{\alpha}, \dot{e} + [x'\cos\alpha - y'\sin\alpha]\dot{\alpha}, 0),$$

$$\vec{n} = (f_{x'}\cos\alpha - f_{y'}\sin\alpha,\ f_{x'}\sin\alpha + f_{y'}\cos\alpha,\ -1)[1 + f_{x'}^2 + f_{y'}^2]^{-1/2}$$

and

$$V_n = [\dot{\bar{x}}_o f_{x'} - \dot{\bar{x}}_o \alpha f_{y'} + e f_{y'} + (x' f_{y'} - y' f_{x'})\alpha][1 + f_{x'}^2 + f_{y'}^2]^{-1/2} + O(\epsilon^3)$$

$$= \dot{\bar{x}}_o f_{x'} [1 + f_{x'}^2 + f_{y'}^2]^{-1/2} + O(\epsilon^2).$$

We may then finally put (56) in the form

$$\frac{d}{d\tau} \frac{1}{2\pi} \iint_{S_o} \dot{\bar{x}}_o(\tau) f_{\xi'}(\xi', \eta') G(\bar{x}, \bar{y}, \bar{z}; \dot{x}_o(\tau) + \xi', \eta', 0; t-\tau)\, d\xi' d\eta'$$

$$+ \frac{1}{2\pi} \iint_{S_o} \dot{\bar{x}}_o(\tau) f_{\xi'} G_t\, d\xi' d\eta' - \frac{1}{2\pi} \iint_{S_o} \dot{\bar{x}}_o(\tau) [f_{\xi'\xi'} G + f_{\xi'} G_{\xi}] d\xi' d\eta'$$

$$+ O(\epsilon^2). \tag{60}$$

We are now ready to construct the function $\phi(\bar{x}, \bar{y}, \bar{z}, t)$. The function $\phi_t(\bar{x}, \bar{y}, \bar{z}, t)$ is the sum of (55) and (60). Hence, after integrating with respect to τ from $\tau = 0$ to $\tau = t$ we find

$$\phi(\bar{x}, \bar{y}, \bar{z}, t) = \phi(\bar{x}, \bar{y}, \bar{z}, 0) + \frac{1}{4\pi} \iint_{S_{FP}(\tau)} [\bar{Y}_t(\bar{\xi}, \bar{\zeta}, t) G(\bar{x}, \bar{y}, \bar{z}; \bar{\xi}, 0, \bar{\zeta}; 0) + \bar{Y} G_t] d\bar{\xi} d\bar{\zeta}$$

$$- \frac{1}{4\pi} \iint_{S_{FP}(o)} [\bar{Y}_t(\bar{\xi}, \bar{\zeta}, 0) G(\bar{x}, \bar{y}, \bar{z}; \bar{\xi}, 0, \bar{\zeta}; t) + \bar{Y} G_t] d\bar{\xi} d\bar{\zeta}$$

$$+ \frac{1}{2\pi} \dot{\bar{x}}_o(t) \iint_{S_o} f_{\xi'}(\xi', \eta') G(\bar{x}, \bar{y}, \bar{z}; \bar{x}_o(t) + \xi', \eta', 0; 0)\, d\xi' d\eta'$$

$$- \frac{1}{2\pi} \dot{\bar{x}}_o(o) \iint_{S_o} f_{\xi'} G(\bar{x}, \bar{y}, \bar{z}; \bar{x}_o(o) + \xi', \eta', o; t)\, d\xi' d\eta'$$

$$+ \frac{1}{2\pi} \int_o^t d\tau\, \dot{\bar{x}}_o(\tau) \iint_{S_o} f_{\xi'} G_t(x, y, z; x\ (\tau) + \xi', \eta', 0; t-\tau) d\xi' d\eta'$$

$$- \frac{1}{2\pi} \int_o^t d\tau\, \dot{\bar{x}}_o^2(\tau) \iint_{S_o} [f_{\xi'\xi'} G + f_{\xi'} G_{\xi}] d\xi' d\eta' + O(\epsilon^2).$$

Because of the initial conditions for G the first integral will vanish. The second and fourth will vanish because initially the water was still

and the ship at rest. For the same reason we may take $\phi(\bar{x},\bar{y},\bar{z},0) = 0$. Hence we finally have

$$\phi(\bar{x},\bar{y},\bar{z},t) = \frac{1}{2\pi}\dot{x}_o(t) \int\int_{S_o} f_{\xi'}(\xi',\eta')G(\bar{x},\bar{y},\bar{z};\bar{x}_o(t)+\xi',\eta',0;0)d\xi'd\eta'$$

$$+\frac{1}{2\pi}\int_0^t d\tau\, \dot{x}_o(\tau)\int\int_{S_o} f_{\xi'}G_t(\bar{x},\bar{y},\bar{z};\bar{x}_o(\tau)+\tau',\pi',0;t-\tau)d\xi'd\eta'$$

$$-\frac{1}{2\pi}\int_0^t d\tau\, \dot{x}_o^2(\tau)\int\int_{S_o} [f_{\xi'\xi'}G + f_{\xi'}G_{\xi}]d\xi'd\eta' + O(\epsilon^2). \tag{61}$$

To the order indicated this will satisfy both the kinematic boundary condition and the initial conditions on the ship hull and both kinematic and dynamic boundary conditions and initial conditions on the free surface.

The last integral in (61) can also be expressed as the following contour integral:

$$-\frac{1}{2\pi}\int_0^t d\tau\, \dot{x}_o^2(\tau)\int_{C_o} f_{\xi'}G\,\tilde{n}_1 ds, \tag{62}$$

where $\tilde{n}_1$ is the x'-component of the external normal to the boundary C_o of S_o. It is evidently zero except along the stem and stern.

There remain to be considered the dynamical boundary conditions on the ship. From (4) the pressure is given by

$$p(\bar{x},\bar{y},\bar{z},t) = -\rho g\bar{y} - \rho\phi_t(\bar{x},\bar{y},\bar{z},t) + O(\epsilon^2)$$

$$= -\rho g\bar{y} - \frac{1}{2\pi}\rho\,\ddot{x}_o(t)\iint_{S_o} f_{\xi'}G(x,y,z;x_o(t)+\xi',\eta',o)d\xi'd\eta'$$

$$+\frac{1}{2\pi}\rho\,\dot{x}^2(t)\iint_{S_o} f_{\xi'\xi'}G\, d\xi'd\eta'$$

$$-\frac{1}{2\pi}\rho\int_0^t d\tau\,\dot{x}_o(\tau)\iint_{S_o} f_{\xi'}G_{tt}(x,y,z;x_o(\tau)+\xi',\eta',o;t-\tau)d\xi'd\eta'$$

$$+\frac{1}{2\pi}\rho\int_0^t d\tau\,\dot{x}_o^2(\tau)\iint_{S_o}[f_{\xi'\xi'}G_t + f_{\xi'}G_{\xi t}]\, d\xi'd\eta'$$

$$+ O(\epsilon^2). \tag{63}$$

Since we wish to integrate this over the surface of the ship, following (9), it is convenient to write

$$p(\bar{x},\bar{y},\bar{z},t) = p(\bar{x}_o(t) + x'\cos\alpha - y'\sin\alpha, e + x'\sin\alpha + y'\cos\alpha, f, t)$$

$$= p(\bar{x}_o(t) + x', y', o, t) + O(\epsilon). \qquad (64)$$

We note now that $\bar{x}$ occurs in (63) only in G, and here only in the combination $\bar{x} - \bar{\xi} = \bar{x}_o(t) - \bar{x}_o(\tau) + x' - \xi' + O(\epsilon)$, so that the $\bar{x}_o$'s drop out in the first two integtals, but are retained in the last two.

In the computation of the force and moment components in (9) the hydrostatic part of p contributes, up to $O(\epsilon^2)$, only to the same extent as in (45), and in particular not to F_x. For this we find

$$F_x = \frac{1}{\pi}\rho\,\ddot{\bar{x}}_o(t) \iint_{S_o} dxdy \iint_{S_o} d\xi d\eta\; f_x(x,y) f_\xi(\xi,\eta) G(x,y,0;\xi,\eta,0;0)$$

$$+ \frac{1}{\pi}\rho\,\dot{\bar{x}}_o^2(t) \iint_{S_o} dxdy \iint_{S_o} d\xi d\eta\; f_x(x,y) f_{\xi\xi}(\xi,\eta) G(x,y,0;\xi,\eta,0;0)$$

$$- \frac{1}{\pi}\rho \int_0^t d\tau\, \dot{\bar{x}}_o(\tau) \iint_{S_o} dxdy \iint_{S_o} d\xi d\eta\; f_x f_\xi G_{tt}(x_o(t) + x, y, 0; x_o(\tau) + \xi, \eta, 0; t-\tau)$$

$$+ \frac{1}{\pi}\rho \int_0^t d\tau\, \dot{\bar{x}}_o^2(\tau) \iint_{S_o} dxdy \iint_{S_o} d\xi d\eta\; f_x [f_{\xi\xi} G_t + f_\xi G_{\xi t}]$$

$$+ O(\epsilon^3). \qquad (65)$$

The first equation of motion in (10) can then be expressed as follows:

$$m(\ddot{\bar{x}}_o - y_G'\ddot{\alpha}) = F_x + T(t) + O(\epsilon^3) \quad . \qquad (66)$$

The equation (66) is really a very complicated integro-differential equation for $\bar{x}_o(t)$, for $\bar{x}_o$ occurs in each term of (65). It is, in fact, incomplete without the other equations in (10), for $\bar{x}_o$, α and e are evidently intertwined. We shall not, however, deal with these equations and hence shall not write them out. However, it is these equations which tell us that α and e are each $O(\epsilon)$.

With respect to (65) there are two points of interest. The first is that the second and fourth integrals do not occur in the expression usually given for F_x. The customary derivation linearizes the boundary conditions at the outset and then finds a solution to the newly set problem. These terms would inevitably be missed by this procedure. The second point concerns limitations inherent in the derivation of (65). In the derivation of the first formula in (45) we were able to make use of the velocity potential far from the ship and thus avoid certain restrictive assumptions about its shape. In the present case this is evidently not possible and we must be prepared to approximate the values of p on the ship hull. This is likely to restrict the applicability of (65) to a narrower class of ship form.

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DISCUSSION

by H. Maruo

I appreciate Professor Wehausen's elaborate analysis of the thin ship theory. He refers the short note about the validity of the thin ship theory which appears in my paper of 1962. I am unhappy, however, to find out that Professor Wehausen's discussion is confined in the first order linearized term. The thin ship theory is complete and self-consistent as a linearized theory, so that nothing can be added as far as the linearized theory is concerned. Unfortunately, the results obtained from the thin ship theory often shows only a poor agreement with the measurement. This defect is due to the limitation of the linear term. The procedure of linearization assumes the expansion of the type

$$\phi = \varepsilon\phi_1 + \varepsilon^2\phi_2 + \varepsilon^3\phi_3 + \ldots..$$

However there is no justification about the possibility of the above expansion. One may think of Weierstrass' theorem that the expansion is possible for sufficient small values of ε, but to apply Weierstrass' theory, ϕ should be regular at the origin. However, the only thing we know is that ϕ becomes zero when ε is made zero. This is not sufficient to make ϕ regular. There is no difficulty to obtain the first term ϕ, that is the linear term, but it can claim its validity only at a vanishing ε.

The discrepancies between computed and measured values indicate that the other terms still remain comparable with the first. Even though the possibility of the expansion be assumed, the radius of convergence may be finite. It depends upon the Froude number, and it may become very small at a certain range of Froude number. In such an occasion, the thin ship theory can be valid only for a ship as thin as a paper. As the wave resistance theory is not a problem of mathematics but is a problem of naval architecture, the problem which we are concerned about is not what type of the ship fits the theory but is what type of the theory fits the ship. The starting point is a ship form which is definitely specified. If we talk about a thin ship, it may have a small beam length ratio, say 1/30 or 1/40. Even in such a case, there is no justification about the condition $\varepsilon\phi_1 \gg \varepsilon^2\phi_2$ for all Froude number ranges. On the contrary, we have a reason to suspect the possibility of $\varepsilon\phi_1 < \varepsilon^2\phi_2$, especially at a low Froude number for which wave number gb/U^2 is large. The validity and limitation of the

thin ship theory concerns itself with the higher order terms. We cannot confine ourselves in the linear term when discussing the limitation of the thin ship theory.

AUTHOR'S REPLY

by J. V. Wehausen

I thought that in a way I was addressing myself to just the problem which Professor Maruo raises, although I can't claim to have been very successful. Formula (30) is an "exact" formula, independent of any assumptions about the ship form (except that it allows application of Green's Theorem) or about the free surface (except that it be representable as a single-valued function $y = Y(x,z)$. There is, of course, the underlying assumption of an inviscid fluid and irrotational flow. My original aim went well beyond Professor Maruo's comparison of $\varepsilon\phi$, with $\varepsilon^2\phi_2$. I had hoped to be able to compare the right-hand side of (30) with the potential obtained from the usual linearized boundary conditions, i.e., (40), and to find numerical estimates in terms of ship form for the difference, i.e., for the term denoted by $O(\varepsilon^2)$ in (40). Unfortunately, I was not able to complete this before the Symposium and was forced back to the $O(\varepsilon^2)$ type of estimate. Even so, a convergent power-series expansion in ε was not assumed. The assumption of (42) and the subsequent computation of (43) was more by way of illustration to show that (30) could also be put to this use.

Professor Maruo's statement that there is "no justification" for assuming such an expansion is certainly correct in so far as no one has proved its convergence for this case. However, convergence proofs (or their equivalent) are available for some two-dimensional motions, and even ones with submerged obstacles. On the basis of these facts, it does not seem totally unjustified to conjecture that such a power series does converge, at least in some region excluding the ship.

I should also like to debate, although not very heatedly, Professor Maruo's contention that nothing can be added to the linearized theory. It seems to me that, insofar as the neglected terms have been explicitly displayed, something has been added. Furthermore, this approach to linearization (which, by the way, is not new; it has been used, for example, by Moiseev and by Newman) has, in fact, given a result in the unsteady case which differs from that obtained from the

usual procedure of assuming an expansion in powers of ε and formulating a sequenc f linearized boundary value problems for each ϕ_n. In connection with his latter procedure I should like to call attention to an interesting paper of V. G. Sizov [On the theory of wave resistance of a ship on calm water, Izvestiya Akad. Nauk SSSR, Otdel. Tekhn. Nauk, Mekh. i Mashinostr. 1961, no. 1, 75-85] in which both ϕ_2 and the second order correction to the resistance are explicitly given for steady motion.

Professor Maruo's statement that we should be concerned with what type of theory fits the ship and not vice versa cannot be debated and is a slogan we can all take to heart. Ships are neither "thin", "flat", nor "slender" and do not sail in inviscid fluids; one should constantly keep this in mind. However, once one has made certain initial decisions concerning a fluid model and a method of approximation, what remains *is* essentially a mathematical problem and should be treated as such. It seems to me that the real import of Professor Maruo's caveat is that we should consider carefully the limits of applicability of each assumption and approximation method before investing a large amount of effort in computational effort. I had hoped to do this.

EXPRESSIONS FOR THE EVALUATION OF WAVE-RESISTANCE FOR POLYNOMIAL CENTERPLANE SINGULARITY DISTRIBUTIONS

F. C. Michelsen

ABSTRACT

General formulae for the evaluation of wave-resistance coefficients which are based upon the Birkhoff-Kotik transformation of Michell's Integral are presented. These formulae are valid for the case of a polynomial distribution of singularities defined over a rectangular region of the center-plane of symmetry. No restrictions are imposed upon the singularity distribution other than that it is continuous and has continuous first derivatives.

The Birkhoff-Kotik Transformation separates the integrand of the Michell Integral into two parts; the Hull Function and the Michell Function. The Hull Function contains all properties of the singularity distribution and is a function of this distribution only. The significance of the formulae presented is, therefore, that the contribution to the wave-resistance resulting from each individual term of the Hull Function polynomial can be independently evaluated. Once these are evaluated over a range of Froude numbers it becomes a comparatively simple matter to compute the wave-resistance for any admissable distribution function. Furthermore, because of a linear superposition, the given expressions for the wave-resistance should provide a method of formulating the necessary equations for the determination of singularity distributions of minimum wave-resistance satisfying prescribed conditions.

1. Introduction

The condition introduced introduced by Michell(1) that the boundary conditions of the hull surface was to be satisfied on the center-plane of symmetry imposed a serious restriction on the linear theory of wave-resistance. This condition made in exact only for ships of zero beam and the theory was therefore appropriately named a thin ship theory. Early experimentation(2,3,4,5) was aimed at determining the magnitude of the error involved when applying the Michell theory to finite beam ships, but the purely experimental evaluation of such an error did not produce reliable results because all that could be measured experimentally at that time was the total resistance of the models being tested. Thus, the experimenters were faced with the problem of deducing from the measured total resistance the magnitude of the wave-resistance, an immense task indeed especially in view of the lack of clear definitions that exist even today. It is clear, therefore, that a theoretical approach to the problem of the hull boundary conditions for finite beam ships in conjunction with experimental verification should prove to be more fruitful than a purely experimental study. In applying the theory a positive approach should be taken, however, i.e., instead of attempting to evaluate the error involved by the application of the thin ship theory to the case of finite beam, we should try to determine to what extent modifications will have to be made to the functions describing the hull characteristics in the wave resistance integrals. In so doing a choice of two methods of approach are available, the direct and the inverse. The direct method requires that a hull characteristics function be found when the hull shape is pre-described. The inverse method, on the other hand, requires that the hull shape be found when the hull characteristics function is pre-described. In the language of the hydrodynamicist, the hull characteristics function is thoughtof as being a function defining a distribution of singularities. Thus, the inverse method consists of determining the closing streamlines, which define the hull shape, for a given singularity distribution and speed.

In his work Inui(6) has applied the inverse method with great success. Although he neglected the singularity system introduced to satisfy free surface conditions, he did clearly demonstrate that the thin ship theory should not be applied to finite beam hull forms. In regard to the free surface conditions, Korvin-Kroukovsky(7) has indicated that the linear velocity potential is accurate to a second order for the case of a two-dimensional train of waves. There is every reason to believe that this is also true for a three-dimensional Kelvin wave system which is composed of infinitesimal two-dimensional waves. This does not mean that the free surface can be neglected in the determination of the hull surface predicted by the linear theory of wave resistance. In fact, it has long been recognized that one of the most difficult problems facing

us is that of satisfying the boundary conditions on the hull in the vicinity of the free surface. The method of streamline tracing employed by Inui should solve this problem, however, once it is made to include the total velocity potential.

Simultaneously with the efforts made towards improving the formulation of the boundary conditions, it is important that work also be continued on improving the mathematical tools necessary for the evaluation of the wave-resistance integrals resulting from the linear theory. The importance of this work is emphasized by the extensive contributions made by Weinblum(8) in this area.

Onesignificant limitation of Weinblum's work is that it admits only special forms of the singularity distribution, in particular it must be describable as a product of two functions each dependent upon only one of the two space coordinates of the center plane. This limitation makes it impossible to locate sinks in the fore-body, and it results in frame lines that will all belong to approximately the same family of curves. Examples of this type of hull form are given by the C-series and S-series of models traced by Inui. The characteristic rocker bottom exhibited by these models is another feature which has been criticized as unrealistic in representing a practical hull form. Inui has proposed the introduction of a line sink-source distribution along the keel line to avoid the rocker bottom shapes. It will probably be much simpler, however, to consider a sink distribution in the lower parts of the forebody, with a corresponding source distribution in the aftbody, which is an integral part of the general distribution function. Such a singularity distribution may, furthermore, be made to represent hull forms of beam draft ratios greater than two.

In 1954 Birkhoff, Korvin-Kroukovsky and Kotik(9) read before the Society of Naval Architects and Marine Engineers a paper outlining two new transformations of the Michell Integral. The first of these transformations probably attracted the greatest attention. It failed to gain general acceptance as a working tool, however, because the published form required a double numerical integration for the evaluation of the wave-resistance coefficients, an integration which was in addition complicated by the necessity of using several assymptotic expansions and by the existence of a singularity in the integrand. In this paper the author introduces mathematical relationships which make it possible to avoid these difficulties, and thus formulae for the wave-resistance coefficient for a polynomial distribution of singularities on the center-plane which are defined in terms of infinite series of known functions are presented. Frequently integral functions are easier to handle numerically than infinite series. Formulae in terms of integral functions derived for that purpose are, therefore, also presented.

The derivations are, as already stated, based on the Michell Integral of wave-resistance. The author prefers, however, to assume that the functions defining the geometry of the hull is in reality the singularity distribution function defined on the center plane, and it will, therefore, be referred to as such. The relationship between the singularity distribution and the hull configuration required to satisfy the hull boundary conditions is not being considered here since this can be dealt with as a separate problem.

2. A Transformation of Michell's Integral

For a ship moving at constant speed V the Michell Integral is given by

$$R_w = \frac{4\rho g^2}{\pi V^2} \int_1^\infty (I^2 + J^2) \frac{\lambda^2 d\lambda}{\sqrt{\lambda^2 - 1}} \tag{2.1}$$

where
$$I = \int_0^\infty \int_{-\infty}^\infty f(x,z) \exp(-\frac{\lambda^2 gz}{V^2}) \cos \frac{\lambda gx}{V^2} \, dxdz$$

$$J = \int_0^\infty \int_{-\infty}^\infty f(x,z) \exp(-\frac{\lambda^2 gz}{V^2}) \sin \frac{\lambda gz}{V^2} \, dxdz$$

The function $f(x,z)$ is to be interpreted as being proportional to the singularity distribution.

Except for infinitely deep struts, $f(x,z)$ is defined as non-zero only on a domain

$$S = \{-\frac{L}{2} \le x \le \frac{L}{2} ; \ 0 \le z \le D \}$$

where L = Length of ship

D = Draft of ship .

It is convenient to express the multiple integrals of Equation (2.1) in a non-dimensional form. For this purpose the following

variables are introduced:

$$u = \frac{x}{L} \; ; \quad \hat{u} = \frac{\hat{x}}{L} \quad ; \quad w = \frac{z}{L} \quad ; \quad \hat{w} = \frac{\hat{z}}{L}$$

Furthermore, let $\frac{gL}{v^2} = F = f^{-2}$ where f is the Froude Number. Then from (2.2)

$$R_W = \frac{4\rho F^2 V^2}{\pi} \int_1^\infty \iint_{S'} \iint_{S'} f(Lu, Lw)\, f(L\hat{u}, L\hat{w})\, \frac{\lambda^2}{\sqrt{\lambda^2-1}} \exp\left(-F\lambda^2(w + \hat{w})\right)$$

$$\times \cos\left[\lambda F(u-\hat{u})\right] L^2 du\,dw\,d\hat{u}\,d\hat{w}\,d\lambda$$

$$S' = \left\{-\frac{1}{2} \leq u \leq \frac{1}{2} \; ; \quad 0 \leq w \leq \frac{D}{L}\right\}$$

Substituting $Bh(u,w) = Lf(x,z)$, where 2B is to be a Michell approximation the beam of the ship, it follows that

$$R_W = \frac{4\rho F^2 V^2 B^2}{\pi} \int_1^\infty \iint_{S'} \iint_{S'} h(u,w)h(\hat{u},\hat{w})\, \frac{\lambda^2}{\sqrt{\lambda^2-1}} \exp\left(-\lambda^2 F(w + \hat{w})\right)$$

$$\times \cos\left[\lambda f(u-\hat{u})\right] du\,d\hat{u}\,dw\,d\hat{w}\,d\lambda \qquad (2.2)$$

In Equation (2.2) let

$$\xi = \hat{u} - u \; ; \quad \zeta = \hat{w} + w \; .$$

Then for u and w constant, we have that

$$d\xi = d\hat{u} \; ; \qquad d\zeta = d\hat{w}$$

and (2.2) becomes

$$C_W = \frac{8F^2}{\pi} \int_{+1}^{-1} du \int_0^{\frac{2D}{L}} dw\, h(u,w) \int_1^\infty d\lambda \int_{-\frac{1}{2}-u}^{\frac{1}{2}-u} d\xi \int_w^{w+\frac{D}{L}} d\zeta\; h(\xi + u, \zeta - w)$$

$$\times \exp\left(-\lambda^2 F \zeta\right) \frac{\lambda^2}{\sqrt{\lambda^2-1}} \quad \cos(\lambda F \xi)$$

If the order of integration is interchanged, C_w can be written

$$C_w = \frac{8F^2}{\pi} \int_{+1}^{-1} d\xi \int_0^{\frac{2D}{L}} d\zeta \left\{ \int_{-\xi-\frac{1}{2}}^{\frac{1}{2}-\xi} du \int_{\zeta}^{\zeta-\frac{D}{L}} dw \; h(u,w) h(\xi + u, \zeta - w) \right\}$$

$$\times \left\{ \int_1^{\infty} e^{-\lambda^2 F\zeta} \cos(\lambda F\xi) \frac{\lambda^2}{\sqrt{\lambda^2-1}} \, d\lambda \right\}$$

or in a more compact form

$$C_w = \frac{8F^2}{\pi} \int_{-1}^{1} d\xi \int_0^{\frac{2D}{L}} d\zeta \; H(\xi,\zeta) \, C(F\xi, F\zeta) \tag{2.3}$$

where

$$H(\xi,\zeta) = \int_{-\xi-\frac{1}{2}}^{\frac{1}{2}-\xi} du \int_{\zeta-\frac{D}{L}}^{\zeta} dw \; h(u,w) \; h(\xi + u, \zeta - w) \tag{2.4}$$

$$C(s,t) = \int_1^{\infty} e^{-t\lambda^2} \cos(s\lambda) \frac{\lambda^2 d\lambda}{\sqrt{\lambda^2-1}} \tag{2.5}$$

Here $H(\xi, \zeta)$ is called the Hull Function and $C(s,t)$ the Michell Function.

Birkhoff and Kotik has shown that Equations (2.2) and (2.3) are equivalent.

3. The Hull Function

From the form of Equation (2.2), it follows that

$$H(-\xi, \xi) = H(\xi, \zeta)$$

Obviously, the Michell Function is symmetric with respect to s. It suffices therefore to consider positive values of ξ only.

Thus, (2.3) can be replaced by

$$c_w = \frac{16F^2}{\pi} \int_0^1 d\xi \int_0^{\frac{2D}{L}} d\zeta \; H(\xi,\zeta)\, c(F\xi, F\zeta) \tag{3.1}$$

Since $h(u,w) = 0$ everywhere outside the region S', the limits of integration of Equation (2.4) can be reduced somewhat. If $\xi \geq 0$, then $h(u,w) = 0$ for $u < -\frac{1}{2}$, and the lower limit on u in (2.4) becomes $(-\frac{1}{2})$. In regard to the limits on w, two cases must be considered.

(i) $\zeta \leq \frac{D}{L}$; the limit $\zeta - \frac{D}{L}$ can be replaced by zero, since $h(u,w) = 0$, $w < 0$.

(ii) $\frac{D}{L} \leq \zeta \leq \frac{2D}{L}$; the upper limit on w can be replaced by $w = \frac{D}{L}$, since $h(u,w) = 0$, $w > \frac{D}{L}$.

The Hull Function may therefore be defined as follows:

$$H(\xi,\zeta) = [H(\xi,\zeta)]_I = \int_{-\frac{1}{2}}^{\frac{1}{2}-\xi} du \int_0^{\zeta} dw \; h(u,w)h(\xi+u,\; \zeta-w) \qquad \xi \geq 0 \;;\; 0 \leq \zeta \leq \frac{D}{L} \tag{3.2}$$

$$= [H(\xi,\zeta)]_{II} = \int_{-\frac{1}{2}}^{\frac{1}{2}-\xi} du \int_{\zeta-\frac{D}{L}}^{\frac{D}{L}} dw \; h(u,w)h(\xi+u,\; \zeta-w) \qquad \xi \geq 0 \;;\; \frac{D}{L} \leq \zeta \leq \frac{2D}{L}$$

These expressions show immediately that $H(\xi, \zeta) = 0$ along the boundaries of S". Furthermore, it is noted that for small values of ζ, $H(\xi, \zeta)$ can be approximated by

$$H(\xi, \zeta) = \zeta \int_{-\frac{1}{2}}^{\frac{1}{2}-\xi} h(u,0)\; h(u,0)\; du$$

Thus $H(\xi,\zeta)$ vanishes at least as fast as a linear function of ζ on $\zeta \to 0$. By means of similar approximations, it can be shown that the Hull Function vanishes like a linear function (or faster) on the

complete boundary of the region S".

$$S'' = \left\{-1 \le \xi \le 1 \; ; \; 0 \le \zeta \le \frac{2D}{L} \right\}$$

Now by Leibniz rule

$$\frac{\partial H(\xi,\zeta)}{\partial \xi} = \int_0^{\zeta} dw \frac{\partial}{\partial \xi} \int_{-\frac{1}{2}}^{\frac{1}{2}-\xi} h(\xi + u, \; \zeta - w) h(u,w) \, du$$

$$= \int_0^{\zeta} dw \left[-h(\tfrac{1}{2}, \zeta - w) h(\tfrac{1}{2} - \xi, w) + \int_{-\frac{1}{2}}^{\frac{1}{2}-\xi} h(u,w) \frac{\partial}{\partial \xi} h(\xi + u, \; \zeta - w) du \right]$$

Also since

$$\frac{\partial}{\partial \xi} h(u + \xi, \; \zeta - w) = \frac{\partial}{\partial u} h\,(u + \xi, \; \zeta - w)$$

it follows that

$$\left. \frac{\partial H(\xi,\zeta)}{\partial \xi} \right|_{\xi = 0^+} = \int_0^{\zeta} dw \left[-h(\tfrac{1}{2}, \zeta - w) h(\tfrac{1}{2}, w) + \int_{-1/2}^{1/2} h(u,w) \frac{\partial}{\partial u} h(u, \zeta - w) \, du \right] \tag{3.3}$$

If one lets $w' = \zeta - w$, (3.3) may be written

$$\left. \frac{\partial H(\xi,\zeta)}{\partial \xi} \right|_{\xi = 0^+} = \int_0^{\zeta} dw' \left[-h(\tfrac{1}{2}, w') h(\tfrac{1}{2}, \; \zeta - w') + \int_{-1/2}^{1/2} h(u, \zeta - w') \frac{\partial}{\partial u} h(u, w') du \right] \tag{3.4}$$

Dropping the prime, we obtain from (3.3) and (3.4)

$$2\left.\frac{\partial H(\xi,\zeta)}{\partial \xi}\right|_{\xi=0^+} = \int_0^{\zeta} dw[-2h(\tfrac{1}{2},w)h(\tfrac{1}{2},\zeta-w)$$

$$+\int_{-1/2}^{1/2} h(u,\zeta-w)\frac{\partial}{\partial u}h(u,w)du + \int_{-1/2}^{1/2} h(u,w)\frac{\partial}{\partial u}h(u,\zeta-w)du]$$

$$= \int_0^{\zeta} dw\{-2\,h(\tfrac{1}{2},w)h(\tfrac{1}{2},\zeta-w) + \int_{-1/2}^{1/2} \frac{\partial}{\partial u}h(u,w)h(u,\zeta-w)\,du\}$$

$$= -\int_0^{\zeta} dw\,\{h(\tfrac{1}{2},w)h(\tfrac{1}{2},\zeta-w) + h(-\tfrac{1}{2},w)h(-\tfrac{1}{2},\zeta-w)\}$$

Should the singularity distribution function at the bow and stern be zero, then

$$\frac{\partial H(\xi,\zeta)}{\partial \xi} = 0 \quad \text{at} \quad \xi = 0^+$$

The Hull Function for polynomial singularity distributions is derived in Appendix 1.

To gain insight into the physical significance of the Hull Function we shall consider a distribution of singularities defined as non-zero on two infinitesimal areas of magnitude Δ, located at (u_1w_1) and (u_2w_2) respectively, and zero elsewhere.

Let $h(u_1w_1) = h_1$ and $h(u_2w_2) = h_2$

Since $H(\xi,\zeta) = \int du \int dw\, h(u,w)h(\xi+u,\zeta-w)$ the Hull Function will be different from zero only for specific values of ξ and ζ. For example,

$$H(0,w_1) = \Delta h_1^{\,2}$$

$$H(0,w_2) = \Delta h_2^{\,2}$$

$$H(u_1-u_2,\ w_1+w_2) = \Delta h_2 h_1$$

$$H(u_2-u_1,\ w_1+w_2) = \Delta h_1 h_2$$

From these expressions, one concludes that $H(0,\zeta)$ represents the total wave resistance of all the infinitesimal elements of the singularity distribution considered as separate singularities and that $H(\xi,\zeta)$, $\xi > 0$, represents interaction effects.

The contribution to wave resistance by the two singularities, considering interference effect only, is given by (2.3). Thus

$$\delta_1 C_w = \frac{8F^2}{\pi} \Delta^2 h_1 h_2 \int_0^\infty e^{-\lambda^2 F(w_1+w_2)} [\cos \lambda F(u_1-u_2) + \cos \lambda F(u_2-u_1)] \frac{\lambda^2 d\lambda}{\sqrt{\lambda^2-1}}$$

$$= \frac{16F^2}{\pi} \Delta^2 h_1 h_2 \int_0^\infty e^{-\lambda^2 F(w_1+w_2)} \cos \lambda F(u_1-u_2) \frac{\lambda^2 d\lambda}{\sqrt{\lambda^2-1}}$$

This expression is the same as that obtained by Michell from the original form of the Michell Integral.

If the two singularities are located on the same vertical line, $u_1 = u_2$ or $\xi = 0$, the interference effect becomes

$$\delta_1 C_w \Big|_{\xi=0} = \frac{16F^2}{\pi} \Delta^2 h_1 h_2 \int_1^\infty e^{-\lambda^2 F(w_1+w_2)} \frac{\lambda^2 d\lambda}{\sqrt{\lambda^2-1}}$$

Michell[1] has shown that

$$\int_1^\infty e^{-a\lambda^2} \frac{\lambda^2 d\lambda}{\sqrt{\lambda^2-1}} = \frac{1}{4} e^{-a/2} [K_o(\tfrac{a}{2}) + K_1(\tfrac{a}{2})] \tag{3.5}$$

where $K_o(\frac{a}{2})$ and $K_1(\frac{a}{2})$ are the modified Bessel functions of the second kind. From this it follows that

$$\delta_1 C_w \Big|_{\xi=0} = \frac{4F^2}{\pi} \Delta^2 h_1 h_2 e^{-F\frac{w_1+w_2}{2}} [K_o(F\frac{w_1+w_2}{2}) + K_1(F\frac{w_1+w_2}{2})]$$

The contribution to wave resistance by the two singularities considered separately is given by

$$\delta 2C_w = \frac{8F^2}{\pi}\Delta^2[h_1{}^2\int_1^\infty e^{-2\lambda^2 Fw_1}\frac{\lambda^2 d\lambda}{\sqrt{\lambda^2-1}} + h_2{}^2\int_1^\infty e^{-2\lambda^2 Fw_2}\frac{\lambda^2 d\lambda}{\sqrt{\lambda^2-1}}]$$

$$= \frac{2F^2}{\pi}\ \Delta^2\ \{h_1{}^2 e^{-Fw_1}[K_0(Fw_1)+K_1(Fw_1)] + h_2{}^2 e^{-Fw_2}[K_0(Fw_2)+K_1(Fw_2)]\}$$

This expression shows that $\delta_2 C_w$ is always positive, and its magnitude depends upon how far the singularities are located below the undisturbed surface and upon their strength.

From what has been said, it becomes apparent that the wave-resistance of a ship is made up of two parts, namely:

a) The sum of the effects of all infinitesemal singularities considered separately; and

b) The sum of interference effects between any two singularities taken in pairs.

It has been shown that the interference effects depend upon the horizontal distance between the singularities and the sum of vertical distances to the undisturbed surface. The horizontal distance is represented by the variable ξ and the sum of vertical distances by the variable ζ. Thus, for a given value of ξ and ζ, the Hull Function $H(\xi,\zeta)$ represents the total interference effects of all singularities taken in pairs, having a horizontal spacing equal to ξ and for which the sum of vertical coordinates is equal to ζ.

The wave-resistance is determined by the products of the functions $H(\xi,\zeta)$ and $C(F\xi,F\zeta)$. Over any region in the (ξ,ζ)-plane where these functions are of equal sign, one may say that interaction effects are detrimental, whereas opposite signs indicate favorable conditions. Since $C(F\xi,F\zeta)$ is completely determined for specific value of F (a given Froude number), the Hull function represents all the wave-resistance characteristics of the ship.

4. The Michell Function

The Michell Function is defined by (2.5) as

$$C(s,t) = \int_1^\infty e^{-t\lambda^2} \cos(s\lambda) \frac{\lambda^2 d\lambda}{\sqrt{\lambda^2-1}}$$

Differentiating under the integral sign, one observes that $C(s,t)$ satisfies the homogeneous heat equation:

$$\frac{\partial^2 u}{\partial s^2} = \frac{\partial u}{\partial t} \; ; \quad t > 0 \; .$$

It is known that

$$\int_1^\infty \cos(x,\lambda) \frac{d\lambda}{\sqrt{\lambda^2-1}} = - \frac{\pi}{2} \; Y_o(x)$$

where $Y_o(x)$ is the Bessel function of the second kind. Furthermore, it can be verified that

$$\lim_{t \to 0^+} \int_1^\infty e^{-\lambda^2 t} \cos(s\lambda) \frac{d\lambda}{\sqrt{\lambda^2-1}} = \int_1^\infty \cos(s\lambda) \frac{d\lambda}{\sqrt{\lambda^2-1}}$$

Hence it follows that

$$\lim_{t \to 0^+} C(s,t) = \frac{\pi}{2} \frac{\partial^2}{\partial s^2} Y_o(s)$$

An expression for $C(0,t)$ is obtained by making use of the relationship (3.5), namely

$$C(0,t) = \frac{e^{-\frac{t}{2}}}{4} \left[K_o\left(\frac{t}{2}\right) + K_1\left(\frac{t}{2}\right)\right]$$

An asymptotic expression for large value of s and fixed t, obtained by J. Kotic and rederived in Reference 10, is given by

$$C(s,t) = \sqrt{\frac{\pi}{2s}} \; e^{-t} \Big\{ \cos\left(s + \frac{\pi}{4}\right) - \frac{1}{s}\left(\frac{7}{8} - t\right) \sin\left(s + \frac{\pi}{4}\right)$$

$$- \frac{1}{s^2}\left(\frac{3}{2} t^2 - \frac{27}{8} t + \frac{57}{128}\right) \cos\left(s + \frac{\pi}{4}\right) + O(s^{-3}) \Big\}$$

It follows from this formula that $C(s,t)$ tends to zero for large values of s and t.

Birkhoff[10] has outlined methods and procedures for obtaining numerical values of the Michell Function. If a formal integration of the product of Hull Function and Michell Function over the region S'' is contemplated, however, where $S'' \left\{0 \leqq \xi \leqq 1 \; ; \; 0 \leqq \zeta \leqq \frac{2D}{L}\right\}$, numerical values are not sufficient. A general expression of the Michell Function must be found. One method of obtaining such an expression is as follows:

In Equation (2.5) let

$$\lambda^2 = \bar{\lambda} + 1 \; ; \quad d\lambda = \frac{d\bar{\lambda}}{2\sqrt{\bar{\lambda} + 1}}$$

so that the Michell Function becomes

$$C(s,t) = \frac{1}{2} e^{-t} \int_0^\infty e^{-t\bar{\lambda}} \cos(s\sqrt{\bar{\lambda} + 1}) \frac{\sqrt{\bar{\lambda} + 1}}{\bar{\lambda}^{1/2}} d\bar{\lambda}$$

$$= \frac{1}{2} e^{-t} L \left\{\cos(s\sqrt{\bar{\lambda} + 1}) \frac{\sqrt{\bar{\lambda} + 1}}{\bar{\lambda}^{1/2}}\right\} \qquad (4.1)$$

where $L\{\;\}$ is the Laplace Transform operator. By Reference 11, the transform of (4.1) exists.

From the relationship

$$J_{-1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \cos z$$

one has that

$$\cos(s\sqrt{\bar{\lambda} + 1}) = \left(\frac{\pi}{2} s\sqrt{\bar{\lambda} + 1}\right)^{1/2} J_{-1/2}(s\sqrt{\bar{\lambda} + 1})$$

Furthermore, it can be shown that (Appendix II)

$$\left(\frac{s}{2}\right)^{\nu} \left(\frac{1}{2} s\sqrt{\bar{\lambda} + 1}\right)^{-\nu} J_\nu(s\sqrt{\bar{\lambda} + 1}) = \sum_{n=0}^{\infty} \frac{J_{\nu+n}(s)\,(-s\lambda)^n}{2^n\, n!}$$

thus

$$\cos(s\sqrt{\bar{\lambda} + 1}) = (\frac{\pi s}{2})^{1/2} \sum_{n=o}^{\infty} \frac{(-1)^n J_{-\frac{1}{2}+n}(s)(s\bar{\lambda})^n}{2^n \; n!} \tag{4.2}$$

Substituting into Equation (4.1)

$$C(s,t) = \frac{1}{2}(\frac{\pi s}{2})^{1/2} e^{-t} \sum_{n=o}^{\infty} \frac{(-1)^n J_{-\frac{1}{2}+n}(s) \; s^n}{2^n \; n!} \int_o^{\infty} e^{-t\bar{\lambda}} \bar{\lambda}^{n-\frac{1}{2}} (1+\bar{\lambda})^{\frac{1}{2}} \, d\bar{\lambda} \tag{4.3}$$

By Reference 12, p. 269

$$\int_o^{\infty} e^{-t\bar{\lambda}} \bar{\lambda}^{n-\frac{1}{2}} (1+\lambda)^{1/2} \, d\bar{\lambda} = \Gamma(n + \frac{1}{2}) \; \Psi(n + \frac{1}{2}, n+2; t) \tag{4.4}$$

where $\Psi(n + \frac{1}{2}), n+2; t)$ is the Confluent Hypergeometric Function. Since (n+2) is always a positive integer, $\Psi(n + \frac{1}{2}, n+2; t)$ is logarithmic near origin.

Using the relationship

$$\Gamma(n + \frac{1}{2}) = \sqrt{\pi} \; (\frac{1}{2})_n$$

where

$$(\frac{1}{2}) = \frac{1}{2}(\frac{1}{2} + 1)(\frac{1}{2} + 2) \; \ldots \; (\frac{1}{2} + n - 1)$$

and substituting (4.4) in (4.3), the expression for the Michell Function becomes

$$C(s,t) = \frac{\pi}{2}(\frac{s}{2})^{1/2} e^{-t} \sum_{n=o}^{\infty} \frac{(-1)^n (\frac{1}{2})_n \; s^n}{2^n n!} [J_{-\frac{1}{2}+n}(s)] \; \Psi(n + \frac{1}{2}, n+2; t] \tag{4.5}$$

5. Wave-Resistance Coefficients

The wave-resistance of a surface ship moving at constant speed on a straight course in water of infinite depth is given by Equation (3.1) with the Hull Function as defined by (3.2) and the Michell Function by (4.5).

It will now be assumed that the Hull Function can in general be expressed as polynomials, i.e.,

$$[H(\xi,\zeta)]_I = \sum_{\alpha,\beta}^{M,N} A^I_{\alpha\beta}\, \xi^\alpha \zeta^\beta$$

$$[H(\xi,\zeta)]_{II} = \sum_{\alpha,\beta}^{M,N} A^{II}_{\alpha\beta}\, \xi^\alpha \zeta^\beta \tag{5.1}$$

where $A^I_{\alpha\beta}$ and $A^{II}_{\alpha\beta}$ are constants and α and β are non-negative integers. Since $H(\xi,\zeta)$ vanishes at least as fast as a linear function as $\zeta \to 0$, one concludes that in $[H(\xi,\zeta)]_I$ $\beta \geqq 1$.

In the following, expressions for the contribution to wave resistance due to a general term of the polynomials of (5.1) will be presented. Depending upon whether the term belongs to the first or second polynomial, this contribution will be defined by $[\Delta^{\alpha\beta}C_w]_I$ and $[\Delta^{\alpha\beta}C_w]_{II}$ respectively.

From Equations (3.1), (4.5), and (5.1), it follows that

$$[\Delta^{\alpha\beta}C_w]_I = \frac{16F^2}{\pi} \int_0^1 d\xi \int_0^{D/L} d\zeta\, A^I_{\alpha\beta}\, \xi^\alpha \zeta^\beta\, \frac{\pi}{2}\left(\frac{\xi F}{2}\right)^{1/2} e^{-\zeta F}$$

$$\left\{ \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{1}{2})_n (F\xi)^n}{2^n n!} \left[J_{-\frac{1}{2}+n}(\xi F)\right] \Psi(n+\tfrac{1}{2}, n+2; F\zeta) \right\} \tag{5.2}$$

Introducing new variables defined by

$F\xi = \bar{\xi}$ and $F\zeta = \bar{\zeta}$, and dropping the bar

Equation (5.2) becomes

$$[\Delta^{\alpha\beta}C_w]_I = \frac{4\sqrt{2}\, A^I_{\alpha\beta}}{F^{\alpha+\beta}} \int_0^F d\xi \int_0^{FD/L} d\zeta \left\{ \xi^{\alpha+\frac{1}{2}} \zeta^\beta e^{-\zeta} \right\}$$

$$\left\{ \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{1}{2})_n (\xi)^n}{2^n n!} \left[J_{-\frac{1}{2}+n}(\xi)\right] \Psi(n+\tfrac{1}{2}, n+2; \zeta) \right\} \tag{5.3}$$

Since the logarithmic case of the Confluent Hypergeometric Function has a singularity at $\zeta = 0$, consider

$$I_1 = \lim_{\epsilon \to 0^+} \int_{\epsilon}^{FD/L} e^{-\xi} \zeta^{\beta} \Psi(n + \tfrac{1}{2}, n+2; \zeta) d\zeta \tag{5.4}$$

Now from Reference 12

$$\frac{d}{d\zeta} [e^{-\zeta} \Psi(n + \tfrac{1}{2}, n+1; \zeta)] = -e^{-\zeta} \Psi(n + \tfrac{1}{2}, n+2; \zeta) \tag{5.5}$$

If (5.4) is integrated by parts, one obtains

$$I_1 = \lim_{\epsilon \to 0^+} \{-\zeta^{\beta} e^{-\zeta} \Psi(n + \tfrac{1}{2}, n+1; \zeta) \Big|_{\epsilon}^{FD/L}$$

$$+ \beta \int_{\epsilon}^{FD/L} \zeta^{\beta-1} e^{-\zeta} \Psi(n + \tfrac{1}{2}, n+1; \zeta) d\zeta\}$$

so that by iteration,

$$I_1 = \lim_{\epsilon \to 0^+} \{-e^{-\zeta} \sum_{\bar{k}=o}^{\beta} (-\beta)_{\bar{k}} (-1)^{-\bar{k}} \zeta^{\beta-\bar{k}} \Psi(n + \tfrac{1}{2}, n+1-\bar{k}; \zeta) \Big|_{\epsilon}^{FD/L} \} \tag{5.6}$$

Substituting (5.6) in (5.3), it follows that

$$[\Delta^{\alpha\beta} C_w]_I = \frac{4\sqrt{2}\, A^I_{\alpha\beta}}{F^{\alpha+\beta}} [- \int_0^F d\xi \{\xi^{\alpha+\frac{1}{2}} \sum_{n=o}^{\infty} \frac{(-1)^n (\frac{1}{2})_n \xi^n}{2^n n!} [J_{-\frac{1}{2}+n}(\xi)]\}$$

$$\{e^{-\frac{FD}{L}} \sum_{\bar{k}=o}^{\beta} (-\beta)_{\bar{k}} (\tfrac{FD}{L})^{\beta-\bar{k}} \Psi(n + \tfrac{1}{2}, n+1-\bar{k}; \tfrac{FD}{L})\}$$

$$+ \lim_{\epsilon \to 0^+} \int_0^F d\xi \{\xi^{\alpha+\frac{1}{2}} \sum_{n=o}^{\infty} \frac{(-1)^n (\frac{1}{2})_n \xi^n}{2^n n!} [J_{-\frac{1}{2}+n}(\xi)]$$

$$\{e^{-\epsilon} \sum_{\bar{k}=o}^{\beta} (-\beta)_{\bar{k}} (\epsilon)^{\beta-\bar{k}} \Psi(n + \tfrac{1}{2}, n+1-\bar{k}; \epsilon)\} \tag{5.7}$$

If the integral representation of the Confluent Hypergeometric Function and relationship (a) of Appendix II is used, the second integral in (5.7) becomes

$$I_2 = \lim_{\epsilon \to 0^+} \left(\frac{2}{\pi}\right)^{1\ 2} \int_0^F d\xi \left\{\xi^{\alpha} e^{-\epsilon} \sum_{\bar{k}=0}^{\beta} (-\beta)_{\bar{k}} (-1)^{\bar{k}} \epsilon^{\beta-\bar{k}} \times\right.$$

$$\int_0^\infty e^{-\epsilon\bar{\lambda}} \cos(\xi \sqrt{\bar{\lambda} + 1})\, \frac{d\bar{x}}{\bar{\lambda}^{-1/2}(\bar{\lambda} + 1)^{k + \frac{1}{2}}}$$

In the limit as $\epsilon \to 0^+$, I_2 reduces to

$$I_2 = \left(\frac{2}{\pi}\right)^{1/2} \beta! \int_0^F \xi^{\alpha} d\xi \int_0^\infty \cos(\xi \sqrt{\bar{\lambda} + 1})\, \frac{d\bar{\lambda}}{\bar{\lambda}^{1/2}(\bar{\lambda} + 1)^{\beta + \frac{1}{2}}}$$

$$= 2(-1)^{\beta}\left(\frac{2}{\pi}\right)^{1/2} \beta! \int_0^F \xi^{\alpha} P_{2(\beta-1)+1}(\xi)\, d\xi \tag{5.8}$$

where $P_{2(\beta-1)+1}(\xi)$ is the Havelock P-function defined by

$$P_{2(\beta-1)+1}(\xi) = (-1)^{\beta} \int_0^{\pi/2} \cos^{2(\beta-1)+1}\Theta \cos(\xi \sec\Theta)\, d\Theta \tag{5.9}$$

Similarly

$$P_{2\beta}(\xi) = (-1)^{\beta} \int_0^{\pi/2} \cos^{2\beta}\Theta \sin(\xi \sec\Theta)\, d\Theta \tag{5.10}$$

Since

$$Y_0(\xi) = -\frac{1}{\pi} \int_0^{\pi/2} \sec\Theta \cos(\xi \sec\Theta)\, d\Theta \tag{5.11}$$

it follows that

$$\frac{d^{2\beta}}{d\xi^{2\beta}} [P_{2(\beta-1)+1}(\xi)] = Y_0(\xi) \tag{5.12}$$

It can readily be shown that

$$I_2 = \frac{2}{\pi}(-1)^{\beta+1}\beta!\,\sqrt{2\pi}\,[\sum_{n=o}^{\infty}(-1)^n \frac{(\frac{1}{2})^{2n}F^{2n+2\beta+\alpha+1}}{(2n+2\beta+\alpha+1)(2n+1)_{2\beta}(n!)^2}\{\gamma + \log F$$

$$-(\frac{1}{2n+2\beta+\alpha+1} + \frac{1}{2n+1} + \frac{1}{2n+2} + \ldots + \frac{1}{2n+2\beta})\}$$

$$-\sum_{n=1}^{\infty}\frac{(-1)^n(\frac{1}{2})^{2n}F^{2n+2\beta+\alpha+1}}{(2n+2\beta+\alpha+1)(2n+1)_{2\beta}(n!)^2}(1 + \frac{1}{2} + \ldots + \frac{1}{n})]$$

$$+\sum_{n=o}^{\beta-1}\frac{(-1)^n F^{2n+\alpha+1}}{(2n+\alpha+1)(2n)!}(\frac{2}{\pi})^{1/2}\beta!\sum_{k=o}^{\beta-1-n}\frac{(-1)^k(\beta-1-n)!}{(2k+1)k!(\beta-1-n-k)!} \qquad (5.13)$$

However, Equation (5.13) is not very convenient to use for large values of F, i.e., low Froude numbers. For numerical calculations we shall therefore be in need of another expression. Integrating repeatedly (5.8) by parts with respect to ξ it follows that

$$\bar{I}_2 = \frac{I_2}{F^{\alpha+\beta}} = (\frac{2}{\pi})^{1/2}\beta!F^{-\beta}\left\{2(-1)^{\beta}[\sum_{n=o}^{[\alpha/2]}(-\alpha)_{2n}F^{-2n}P_{2(\beta+n)}(F)\right.$$

$$+\sum_{n=1}^{[\frac{\alpha+1}{2}]}(-\alpha)_{2n-1}F^{-(2n-1)}P_{2(\beta+n-1)+1}(F)]$$

$$\left.+\alpha!(-1)^{\frac{\alpha+1}{2}}F^{-\alpha}\int_o^{\infty}\frac{d\bar{\lambda}}{\bar{\lambda}^{1/2}(\bar{\lambda}+1)^{(\beta+\frac{1}{2}+\frac{\alpha+1}{2})}}\right\} \qquad (5.14)$$

The last integral is the Eulerian integral of the first kind. It is to be set equal to zero for α an even integer. A simple reduction will show that

$$\int_o^{\infty}\frac{d\bar{\lambda}}{\bar{\lambda}^{1/2}(\bar{\lambda}+1)^{(\beta+\frac{1}{2}+\frac{\alpha+1}{2})}} = \frac{\Gamma(\frac{1}{2})\Gamma(\beta+\frac{\alpha+1}{2})}{\Gamma(\frac{1}{2}+\beta+\frac{\alpha+1}{2})} = \frac{(\beta-1+\frac{\alpha+1}{2})!}{(\frac{1}{2})_{\beta+\frac{\alpha+1}{2}}}$$

Thus

$$\bar{I}_2 = (\frac{2}{\pi})^{1/2}\beta!F^{-\beta}\{2(-1)^{\beta}[(\sum_{n=o}^{\alpha}(-\alpha)_n F^{-n}P_{2\beta+n}(F))]$$

$$+\alpha!(-1)^{\frac{\alpha+1}{2}}F^{-\alpha}\frac{(\beta-1+\frac{\alpha+1}{2})}{(\frac{1}{2})_{\beta+\frac{\alpha+1}{2}}}\} \qquad (5.15)$$

Lunde[13] has given asymptotic expressions for several of the P-functions for both large and small values of the argument. For the range of Froude numbers of usual interest, these are not suitable, however, and we are therefore compelled to compute the P-functions numerically from their integral representation. In checking computations the following recurrence relation is useful.

$$kP_k(x) = x[P_{k-1}(x) + P_{k-3}(x)] - (k-1)P_{k-2}(x) \tag{5.16}$$

Returning now to Equation (5.7) it is noted that the first integral is of the type

$$I_3 = \int_0^F \xi^{\alpha + \frac{1}{2} + n} J_{-\frac{1}{2} + n}(\xi)\, d\xi \tag{5.17}$$

Integration by parts gives

$$I_3 = \sum_{k=0}^{\alpha/2} 2^k \left(-\frac{\alpha}{2}\right)_k F^{\alpha + \frac{1}{2} + n - k} J_{\frac{1}{2} + n + k}(F) \tag{5.18}$$

for α an even integer

and

$$I_3 = \sum_{k=0}^{[\alpha/2]} 2^k \left(-\frac{\alpha}{2}\right)_k F^{\alpha + \frac{1}{2} + n - k} J_{\frac{1}{2} + n + k}(F)$$

$$+ 2^{k_1}\left(-\frac{\alpha}{2}\right)_{k_1} 2^{\frac{\alpha}{2} + n - 1} \pi^{\frac{1}{2}} \Gamma\left(\frac{\alpha}{2} + n + \frac{1}{2}\right) F$$

$$\times \left\{ J_{\frac{\alpha}{2} + n + 1}(F) H_{\frac{\alpha}{2} + n}(F) - J_{\frac{\alpha}{2} + n}(F) H_{\frac{\alpha}{2} + n + 1}(F) \right\}$$

$$+ 2^{k_1}\left(-\frac{\alpha}{2}\right)_{k_1} \frac{F^{\frac{\alpha}{2} + n + 1}}{2\left(\frac{\alpha}{2} + n\right) + !} J_{\frac{\alpha}{2} + n}(F) \tag{5.19}$$

for α an odd integer.

$H_\nu(x)$ is the Struve Function and $k_1 = \left[\frac{\alpha + 1}{2}\right]$. For values of F of

common interest it is sufficiently accurate to use the asymptotic expansion

$$I_3 = \sum_{k=0}^{\infty} 2^k \left(-\frac{\alpha}{2}\right)_k F^{\alpha + \frac{1}{2} + n - k} J_{\frac{1}{2} + n + k}(F) \tag{5.20}$$

which is valid for all values of α. From Equation (5.2) and the expressions for I_2 and I_3 one finally obtains

$$[\Delta^{\alpha\beta} C_w]_I = \frac{4\sqrt{2}\, A^{I}_{\alpha\beta}}{F^{\alpha+\beta}} \Big\{ I_2 - \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{1}{2})_n}{2^n\, n!} I_3$$

$$\Big[e^{-\frac{FD}{L}} \sum_{\bar{k}=0}^{\beta} (-\beta)_{\bar{k}} \left(\frac{FD}{L}\right)^{\beta-\bar{k}} \Psi\left(n + \frac{1}{2}, n+1-\bar{k}; \frac{FD}{L}\right)\Big]\Big\} \tag{5.21}$$

By inspection, the corresponding expression for $[\Delta^{\alpha\beta} C_w]_{II}$ becomes

$$[\Delta^{\alpha\beta} C_w]_{II} = \frac{4\sqrt{2}\, A^{II}_{\alpha\beta}}{F^{\alpha+\beta}} \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{1}{2})_n}{2^n\, n!} I_3$$

$$\Big\{\Big[e^{-\frac{FD}{L}} \sum_{\bar{k}=0}^{\beta} (-\beta)_{\bar{k}} (-1)^{\bar{k}} \left(\frac{FD}{L}\right)^{\beta-\bar{k}} \Psi\left(n + \frac{1}{2}, n + 1 - \bar{k}; \frac{FD}{L}\right)\Big]$$

$$- \Big[e^{-\frac{2FD}{L}} \sum_{\bar{k}=0}^{\beta} (-\beta)_{\bar{k}} (-1)^{\bar{k}} \left(\frac{2FD}{L}\right)^{\beta-\bar{k}} \Psi\left(n + \frac{1}{2}, n + 1 - \bar{k}; \frac{2FD}{L}\right)\Big]\Big\} \tag{5.22}$$

Equations (5.21) and (5.22) presents us with somewhat of a dilemma. For small values of F (high speed) I_3 becomes small and approaches zero as $F \to 0$. The Confluent Hypergeometric function, on the other hand, becomes infinite. In the case of large values of F (low speeds) this situation is reversed. We are fortunate to be able to circumvent our difficulties by means of the following mathematical theorems. By Kummer's first theorem.

$$\Psi(a,c;x) = x^{1-c}\Psi(1+a-c, 2-c;x)$$

and then by a multiplication theorem

$$\Psi(a,c;xy) = y^{1-c} \sum_{p=0}^{\infty} \frac{(1+a-c)_p (1-y)^p}{p!} \Psi(a,c-p;x)$$

It can be shown that

$$[\Delta^{\alpha\beta}C_w]_I = \frac{4\sqrt{2}\,A^I_{\alpha\beta}}{F^{\alpha+\beta}}\{I_2 - e^{-\frac{FD}{L}} \sum_{k=0}^{\beta} (-\beta)_k(-1)^k(\frac{FD}{L})^{\beta-k} \tag{5.21a}$$

$$x\,[\sum_{n=0}^{\infty} \frac{(-1)^n(\frac{1}{2})_n}{2^n\,n!}\,\Psi(k+\frac{1}{2};\,k+1-n;\,F) \sum_{p=0}^{n} \frac{2^p(-n)_p}{p!}(1-\frac{D}{L})^p\,\bar{I}_3(n-p,\alpha,F)]\}$$

where $\bar{I}_3\,(n,\alpha,F) = F^{-n}I_3$

A similar expression can be written immediately for $[\Delta^{\alpha\beta}C_w]_{II}$. From the definitions of the functions involved it is readily verified that Equation (5.21a) is equally well suited for both large and small values of F.

Using the integral representation of the confluent hypergeometric function it follows that Equations (5.21) and (5.22) can be written

$$[\Delta^{\alpha\beta}C_w]_I = 4\sqrt{2}\,A^I_{\alpha\beta}\,\{\frac{I_2}{F^{\alpha+\beta}} - \sum_{k=0}^{\infty}\sum_{\bar{k}=0}^{\beta} C(\frac{D}{L},F)_{k\bar{k}}\,N(k,\bar{k},\frac{FD}{L},F)\} \tag{5.23}$$

and

$$[\Delta^{\alpha\beta}C_w]_{II} = 4\sqrt{2}\,A^{II}_{\alpha\beta}\,\{\sum_{k=0}^{\infty}\sum_{\bar{k}=0}^{\beta} [C(\frac{D}{L},F)_{k\bar{k}}N(k,\bar{k},\frac{FD}{L},F)$$

$$- C(\frac{2D}{L},F)_{k\bar{k}}N(k,\bar{k},\frac{2FD}{L},F)]\} \tag{5.24}$$

where

$$C(\frac{D}{L},F)_{k\bar{k}} = 2^k(-\frac{\alpha}{2})_k\,e^{-\frac{FD}{L}}\,F^{\frac{1}{2}-k-\bar{k}}\,(-\beta)_{\bar{k}}(-1)^{\bar{k}}(\frac{D}{L})^{\beta-\bar{k}}\,\pi^{-\frac{1}{2}} \tag{5.25}$$

and

$$N(k,\bar{k};x,y) = \int_0^{\infty} e^{-xt}\,t^{-\frac{1}{2}}\,(1+t)^{-(\bar{k}+\frac{k}{2}+\frac{3}{4})}\,J_{\frac{1}{2}+k}(y\sqrt{1+t})$$

$$= 2\int_0^{\pi/2} e^{-x\tan^2\Theta}\,\cos\Theta^{2\bar{k}+k-\frac{1}{2}}\,J_{\frac{1}{2}+k}(y\sec\Theta)\,d\Theta \tag{5.26}$$

From the recurrence relation of the Bessel Function it follows that

$$N(k,\bar{k};x,y) = \frac{2(\frac{1}{2} + k + 1)}{y} N(k+1,\bar{k};x,y)$$

$$- N(k+2,\bar{k}-1;x,y) \tag{5.27}$$

Repeated use of Bessel Function recurrence relation leads to the expression

$$N(k,\bar{k};x,y) = (\tfrac{1}{2})_k \pi^{-\frac{1}{2}} (\tfrac{2}{y})^{k+\frac{1}{2}} \int_0^\infty e^{-xt} t^{-\frac{1}{2}} (1+t)^{-(k+\bar{k}+1)}$$

$${}_2F_3 \begin{bmatrix} -\frac{k}{2}, \frac{-k+1}{2}; & \\ & -[y^2(1+t)] \\ \frac{1}{2}, -k, \frac{1}{2} - k; & \end{bmatrix} \sin(y\sqrt{1+t})\, dt$$

$$- (\tfrac{3}{2})_{k-1} \pi^{-\frac{1}{2}} (\tfrac{2}{y})^{k-\frac{1}{2}} \int_0^\infty e^{-xt} t^{-\frac{1}{2}} (1+t)^{-(k+\bar{k}+\frac{1}{2})}$$

$${}_2F_3 \begin{bmatrix} -\frac{k+1}{2} \quad \frac{-k+2}{2} & \\ & -[y^2(1+t)] \\ \frac{3}{2}, -k+1, -k + \frac{1}{2}; & \end{bmatrix} \cos(y\sqrt{1+t})\, dt \tag{5.28}$$

For small values of F (high Froude Numbers) the asymptotic expansion of I_3, which is used for odd values of α, is not sufficiently accurate. In the ragne of $F < 20$ we shall, therefore, replace the summation on k by an integration with respect to ξ and write

$$[\Delta^{\alpha\beta} C_w]_I = 4\sqrt{2}\, A^I_{\alpha\beta} \{\frac{I_2}{F^{\alpha+\beta}} - e^{-\frac{FD}{L}} \sum_{\bar{k}=0}^{\beta} (-\beta)_{\bar{k}}\, F^{-\bar{k}} (\tfrac{D}{L})^{\beta-\bar{k}}$$

$$(x)\ \pi^{-\frac{1}{2}} F^{-\alpha} \int_0^F \xi^{\alpha+\frac{1}{2}} N(-1,k;\tfrac{FD}{L},\xi)\, d\xi \tag{5.29}$$

Similarly

$$[\Delta^{\alpha\beta} C_w]_{II} = 4\sqrt{2}\, A^{II}_{\alpha\beta}\, F^{-\alpha} \sum_{\bar{k}=0}^{\beta} (-\beta)_{\bar{k}}\, F^{-\bar{k}} \left(\frac{FD}{L}\right)^{\beta-\bar{k}}$$

$$(x)\ \pi^{-\frac{1}{2}} \left[e^{-\frac{FD}{L}} \int_0^F \xi^{\alpha+\frac{1}{2}} N(-1,\bar{k};\frac{FD}{L},\xi)\, d\xi \right.$$

$$\left. - e^{-\frac{2FD}{L}} \int_0^F \xi^{\alpha+\frac{1}{2}} N(-1,k;\frac{2FD}{L},\xi)\, d\xi \right] \qquad (5.30)$$

Numerical integration of (5.29) and (5.30) can easily be performed with the aid of Simpson's Rule. The total wave-resistance coefficient is obtained by adding the contributions from the individual terms of the Hull Function polynomials, i.e.

$$C_w = \sum_{\alpha,\beta}^{M,N} [\Delta^{\alpha\beta} C_w]_I + \sum_{\alpha,\beta}^{\bar{M},\bar{N}} [\Delta^{\alpha\beta} C_w]_{II} \qquad (5.31)$$

It should be emphasized that Equation (5.31) is not restricted to ship forms symmetrical fore and aft. The expression is, however, only valid for ships represented by a singularity distribution defined over a rectangular longitudinal plane of symmetry.

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APPENDIX I

The Hull Function Polynomial

Let the general form of the non-dimensional singularity distribution be given by

$$h(u,w) = \sum_{m,n}^{M,N} C_{mn} u^m w^n \tag{1.1}$$

The Hull Function then becomes

$$[H(\xi,\zeta)]_I = \sum_{\alpha=0}^{I+1} \sum_{\beta=1}^{J+1} A^I_{\alpha\beta} \xi^\alpha \zeta^\beta$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}-\xi} du \int_0^\zeta dw \left[\sum_{i,j}^{I,J} C_{ij} u^i w^j \right] \left[\sum_{m,n}^{M,N} C_{m,n} (u+\xi)^m (\zeta - w)^n \right]$$

$$0 \leq \zeta \leq \frac{D}{L} \tag{1.2}$$

and

$$[H(\xi,\zeta)]_{II} = \sum_{\alpha=0}^{I+1} \sum_{\beta=0}^{J+1} A^{II}_{\alpha\beta} \xi^\alpha \zeta^\beta$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}-\xi} du \int_{\zeta - \frac{D}{L}}^{D/L} dw \left[\sum_{i,j}^{I,J} C_{ij} u^i w^j \right] \left[\sum_{m,n}^{M,N} C_{mn} (u+\xi)^m (\zeta - w)^n \right]$$

$$\frac{D}{L} \leq \zeta \leq \frac{2D}{L} \tag{1.3}$$

(1.2) and (1.3) are sets of equations in the coefficients of the Hull Functions $A^I_{\alpha\beta}$ and $A^{II}_{\alpha\beta}$ which are obtained by equating coefficients of equal powers of ξ and ζ. Upon integration (1.2) and (1.3) yield

$$[H(\xi,\zeta)]_I = \sum_{\alpha=0}^{I+1} \sum_{\beta=1}^{J+1} A^I_{\alpha\beta} \xi^\alpha \zeta^\beta$$

$$= \sum_{i=0}^{I} \sum_{j=0}^{J} C_{ij} \sum_{\ell=0}^{i+1} \frac{(i+1)!(-1)^\ell (\frac{1}{2})^{i+1-\ell}}{\ell!(i+1-\ell)!} \xi^\ell$$

$$\times \sum_{m=0}^{I} \sum_{n=0}^{J} C_{mn} \frac{m!\xi^m n!\zeta^{n+j+1}}{(i+1)_{m+1}(j+1)_{n+1}} \sum_{k=0}^{m} \frac{(i+1)_k}{k!} (\tfrac{1}{2})^k \xi^{-k}$$

$$+ \sum_{i=0}^{I} \sum_{j=0}^{J} (-1)^i (\tfrac{1}{2})^{i+1} C_{ij} \sum_{m=0}^{I} \sum_{n=0}^{J} C_{mn} \frac{m!\xi^m n!\zeta^{n+j+1}}{(i+1)_{m+1}(j+1)_{n+1}}$$

$$\times \sum_{k=0}^{m} \sum_{p=0}^{k} \frac{(-1)^p (i+1)_k (\frac{1}{2})^p}{p!(k-p)!} \xi^{-p} \qquad (1.4)$$

and

$$[H(\xi,\zeta)]_{II} = \sum_{\alpha=0}^{I+1} \sum_{\beta=0}^{J+1} A^{II}_{\alpha\beta} \xi^\alpha \zeta^\beta$$

$$= \sum_{i=0}^{I} \sum_{j=0}^{J} C_{ij} (\tfrac{D}{L})^{j+1} (-1)^i (\tfrac{1}{2})^{i+1} \sum_{m=0}^{I} \sum_{n=0}^{J} \frac{m!\xi^m n!\zeta^n C_{mn}}{(i+1)_{m+1}(j+1)_{n+1}}$$

$$\left(\sum_{k=0}^{m} \sum_{p=0}^{k} \frac{(-1)^p (i+1)_k (\frac{1}{2})^p}{p!(k-p)!} \xi^{-p} \right) \left(\sum_{\ell=0}^{n} \sum_{q=0}^{\ell} \frac{(-1)^q (j+1)_\ell (\frac{D}{L})^q \zeta^{-q}}{} \right.$$

$$+ \sum_{i=0}^{I} \sum_{j=0}^{J} C_{ij} (\tfrac{D}{L})^{j+1} \sum_{p=0}^{i+1} \frac{(i+1)!(-1)^p (\frac{1}{2})^{i+1-p}}{p!(i+1-p)!} \xi^p$$

$$\times \sum_{m=0}^{I} \sum_{n=0}^{J} \frac{m!\xi^m n!\zeta^n}{(i+1)_{m+1}(j+1)_{n+1}} C_{mn} \left(\sum_{k=0}^{m} \frac{(i+1)_k}{k!} (\tfrac{1}{2})^k \xi^{-k} \right)$$

$$\times \left(\sum_{\ell=0}^{n} \sum_{q=0}^{\ell} \frac{(-1)^q (j+1)_\ell (\frac{D}{L})^q \zeta^{-q}}{q!(\ell-q)!} \right.$$

$$+ \sum_{i=o}^{I} \sum_{j=o}^{J} C_{ij}(-1)^{i+j}(\tfrac{1}{2})^{i+1} \sum_{q=o}^{j+1} \frac{(j+1)!(-1)^q(\frac{D}{L})^{j+1-q}}{q!(j+1-q)!} \xi^q$$

$$x \sum_{m=o}^{I} \sum_{n=o}^{J} C_{mn} \frac{m!\xi^m n!\zeta^n}{(i+1)_{m+1}(j+1)_{n+1}}$$

$$(\sum_{k=o}^{m} \sum_{p=o}^{k} \frac{(-1)^p(i+1)_k(\frac{1}{2})^p}{p!(k-p)!} \xi^{-p})(\sum_{\ell=o}^{n} \frac{(j+1)_\ell}{\ell!} (\tfrac{D}{L})^\ell \zeta^{-\ell})$$

$$+ \sum_{i=o}^{I} \sum_{j=o}^{J} C_{ij}(-1)^j \sum_{p=o}^{i+1} \frac{(i+1)!(-1)^p(\frac{1}{2})^{i+1-p}}{p!(i+1-p)!} \xi^p$$

$$(\sum_{q=o}^{j+1} \frac{(j+1)!(-1)^q(\frac{D}{L})^{j+1-q}}{q!(j+1-q)!} \zeta^q)$$

$$\sum_{m=o}^{I} \sum_{n=o}^{J} C_{mn} \frac{m!\xi^m n!\zeta^n}{(i+1)_{m+1}(j+1)_{n+1}}$$

$$(\sum_{k=o}^{m} \frac{(i+1)_k}{k!} (\tfrac{1}{2})^k \xi^{-k})(\sum_{\ell=o}^{n} \frac{(j+1)_\ell}{\ell!} (\tfrac{D}{L})^\ell \zeta^{-\ell}) \qquad (1.5)$$

Solution of (1.4) and (1.5) is definitely a task for the computer. This solution is greatly simplified, however, whenever

$$h(u,w) = h_1(u)h_2(w) \qquad (1.6)$$

It follows immediately from the definition of the Hull Function that under the condition of (1.6)

$$[H(\xi,\zeta)]_{I,II} = H_1(\xi)\ H_2(\zeta) \qquad (1.7)$$

As an example consider

$$h(u,w) = -\ 8u(1 - \frac{L}{D} W)$$

thus

$$[H(\xi,\zeta)]_I = \int_{-\frac{1}{2}}^{\frac{1}{2}-\xi} \int_0^{\zeta} 8(u+\xi)8u[1 - \frac{L}{D}(\zeta-w)][1 - \frac{L}{D} W] \, du dw$$

$$= 64 \{\frac{1}{6} \xi^3 - \frac{1}{4} \xi + \frac{1}{12}\} \{\frac{1}{6}(\frac{L}{D})^2 \zeta^3 - (\frac{L}{D})\zeta^2 + \zeta\}$$

$$= H_1^I(\xi) \times H_2^{II}(\zeta)$$

where

$$H_1^I(\xi) = 16(\frac{2}{3} \xi^3 - \xi + \frac{1}{3}) \; ; \quad \xi \geq 0$$

$$H_2^{II}(\zeta) = \zeta - \frac{L}{D} \zeta^2 + \frac{1}{6}(\frac{L}{D})^2 \zeta^3 \quad ; \quad 0 \leq \zeta \leq \frac{D}{L}$$

Similarly

$$[H(\xi,\zeta)]_{II} = H_1^{II}(\xi) \int_{\zeta - \frac{D}{L}}^{D/L} [1 - \frac{L}{D}(\zeta-w)][1 - \frac{L}{D} w] \, dw$$

$$= H_1^{II}(\xi)[\frac{4}{3} \frac{D}{L} - 2\zeta + \frac{L}{D} \zeta^2 - \frac{1}{6}(\frac{L}{D})^2 \zeta^3]$$

$$= H_1^{II}(\xi) \; H_2^{II}(\zeta) \; ; \quad \frac{D}{L} \leq \zeta \leq \frac{2D}{L}$$

$$H_1^{II}(\xi) = H_1^I(\xi)$$

A plot of $H_1(\xi)$ and $H_2(\zeta)$ is shown in Figure 1.

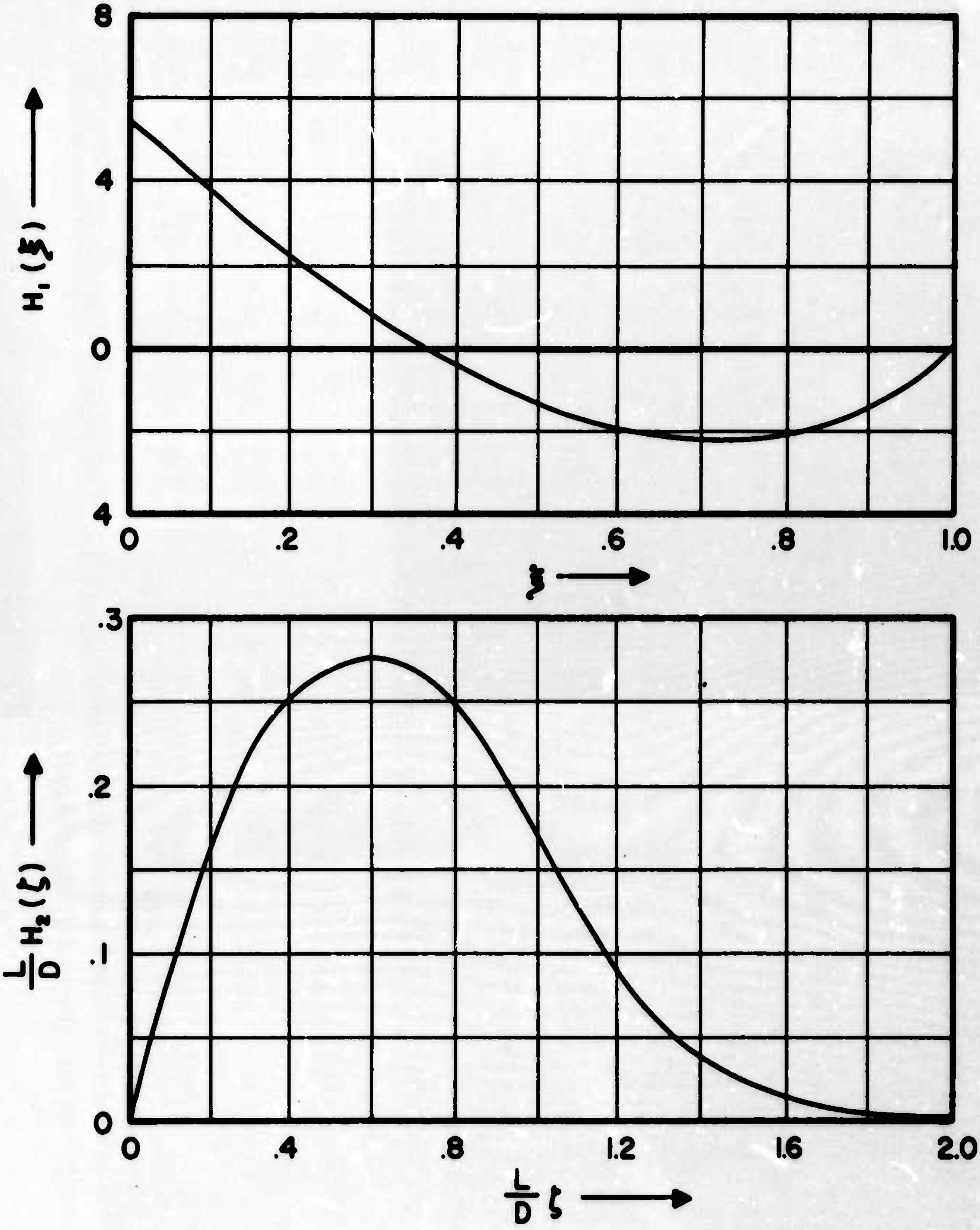

Figure 1 The Hull Function For $h(u,w) = -8u(1 - \frac{L}{D}w)$

APPENDIX II

A Bessel Function Relationship

Consider the generalized hypergeometric function

$$ {}_0F_1\left[\overline{}\,;\ -\frac{x^2(t+1)}{4}\right]_{1+\nu;} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}(t+1)^n}{(1+\nu)_n 2^{2n} n!} $$

$$ = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^n x^{2n} t^{n-k}}{(1+\nu)_n 2^{2n} k!(n-k)!} $$

where the last step follows from the Binomial Theorem. It can be shown that

$$ \sum_{n=0}^{\infty} \sum_{k=0}^{n} A(n,k) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(n+k,k) $$

from which

$$ {}_0F_1\left[\overline{}\,;\ -\frac{x^2(t+1)}{4}\right]_{1+\nu;} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{n+k} x^{2n+2k} t^n}{(1+\nu)_{n+k} 2^{2n+2k} k!n!} $$

$$ = \left(\frac{2}{x}\right)^{\nu} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+n+\nu}\left(-\frac{xt}{2}\right)^n}{(1+\nu)_n(1+\nu+n)_k k!n!2^{n+2k+\nu}} $$

However

$$ (1+\nu)_n = \frac{\Gamma(1+\nu+n)}{\Gamma(1+\nu)} \ ; \ (1+\nu+n)_k = \frac{\Gamma(1+\nu+n+k)}{\Gamma(1+\nu+n)} $$

so that

$$ (1+\nu)_n(1+\nu+n)_k = \frac{\Gamma(1+\nu+n+k)}{\Gamma(1+\nu)} $$

Hence

$$
{}_0F_1\left[\overline{};\atop 1+\nu; \; -\frac{x^2(t+1)}{4}\right]
$$

$$
= \Gamma(1+\nu)\left(\frac{2}{x}\right)^{\nu} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+n+\nu}\left(-\frac{xt}{2}\right)^n}{\Gamma(1+\nu+n+k)k!n!2^{n+2k+\nu}}
$$

$$
= \Gamma(1+\nu)\ \left(\frac{2}{x}\right)^{\nu} \sum_{n=0}^{\infty} \frac{J_{\nu+n}(x)(-1)^n(xt)^n}{2^n n!}
$$

It is also known that

$$
{}_0F_1\left[\overline{};\atop 1+\nu; \; -\frac{x^2(t+1)}{4}\right] = \frac{\Gamma(1+\nu)}{\left(\frac{1}{2}\, x\sqrt{t+1}\right)^{\nu}} J_{\nu}(x\sqrt{t+1})
$$

so that from these two relationships, one obtains

$$
\left(\frac{x}{2}\right)^{\nu}\left(\frac{1}{2}\, x\sqrt{t+1}\right)^{-\nu} J_{\nu}(x\sqrt{t+1}) = \sum_{n=0}^{\infty} \frac{J_{\nu+n}(x)(-xt)^n}{2^n\ n!} \qquad \text{(II.1)}
$$

WAVE-FREE DISTRIBUTIONS AND THEIR APPLICATIONS

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WAVE-FREE DISTRIBUTIONS AND THEIR APPLICATIONS

1. Introduction

The idea of wave free or waveless distributions is a very old one, and we can find it in Lord Kelvin's work in the early history of the water wave research.(1)

Since then, Professors Havelock, Ursell and Maruo have used its idea in their works explicitly or implicitly.(2,3,4)

On the other hand, Professor Inui has proposed theoretically and experimentally the waveless ship a few years ago.(5)

About this time, the author also took up this problem in general and found some of its general properties.(6)

Here, he will show some of the results and applications.(7)

2. Two Dimensional Pressure Distributions(6)

Consider, at first, two dimensional water motions due to a fixed pressure distribution on the surface of a uniform stream with unit velocity and great depth.

Take the origin at the center of the distribution on the undisturbed surface of the stream, with the x-axis horizontally in up-stream direction and the y-axis vertically upwards.

Then, the wave-making resistance of this distribution is given by the formula,

$$R = \rho g^3 |F(g)|^2 \quad , \tag{2.1}$$

with

$$F(p) = \int_{-1}^{1} H(x) \exp.(-ipx)\, dx \ , \tag{2.2}$$

where ρ means the water density, g the gravity constant in this unit system, $|F|$ the absolute value of F and $H(x)$ the water head of the pressure.

Hence, the necessary and sufficient condition for vanishing the wave resistance is clearly

$$F(p) = 0 \quad \text{for} \quad p = g\,, \tag{2.3}$$

that is, $F(p)$ should have a zero value at $p = g$.

It is very easy to seek such function, but here we consider it in a different way. Let us introduce an auxiliary function by the next differential equation

$$(d^2/dx^2 + g^2)\,\sigma(x) = H(x)\ . \tag{2.4}$$

This is a well-known equation, and the function σ is determined uniquely except two boundary conditions when $H(x)$ is given, so that we may use σ instead of H.

Now, putting (2.4) into (2.2) and integrating by parts, we have

$$F(p) = [(d/dx + ip)\,\sigma]\,\exp.(-ipx)\Big|_{x=-1}^{1} + (g^2 - p^2)\int_{-1}^{1}\sigma(x)\,\exp.(-ipx)\,dx, \tag{2.5}$$

so that we may have

$$F(g) = [(d/dx + ip)\,\sigma]\,\exp.(-ipx)\Big|_{x=-1}^{1} \tag{2.6}$$

Thence, if

$$\sigma(\pm 1) = (d/dx)\,\sigma\,(\pm 1) = 0\,, \tag{2.7}$$

$F(g)$ vanishes and the wave resistance vanishes by (2.1).

Thus, to find a wave-free distribution we simply need to find a function σ with the boundary conditions (2.7). Then, the pressure is given by (2.4) and the surface elevation

$$-\eta(x) = g^2\sigma(x) + (g/\pi)\int_{-1}^{1}\frac{(d/dx')\sigma(x')}{x - x'}\,dx'\ . \tag{2.8}$$

It is noteworthy that this surface elevation is symmetric when σ is symmetric. For example, let us consider

$$\sigma(x) = (1/g^2)(1 - x^2)^2\,, \tag{2.9}$$

Then, we have

$$H(x) = (1 - x^2)^2 + 4(3x^2 - 1)/g^2 , \tag{2.10}$$

$$-\eta(x) = (1-x^2)^2 + [8(2/3-x^2) + 4x(1-x^2)\log(1-x/1+x)]/\pi g , \tag{2.11}$$

$$\int_{-1}^{1} H(x)dx = 16/15, \quad -\int_{-1}^{1} \eta(x)dx = 16/15 + 8/(3\pi g) , \tag{2.12}$$

We see these curves in Figure 1.

In low speed, we may notice the next approximation except near both ends.

$$H(x) \doteqdot -\eta(x) \doteqdot g^2\sigma(x) . \tag{2.13}$$

3. Three Dimensional Case I[6]

Consider the three dimensional flow and take the origin at midship on the undisturbed stream, the x-axis horizontally in upstream direction and the z-axis vertically upwards.

The wave-resistance of the Michell-Havelock type ship is given as

$$R = (\rho g^4/\pi) \int_0^{\pi/2} |F(g\sec^2\theta, \theta)|^2 \sec^5\theta \, d\theta, \tag{3.1}$$

with

$$F(k,\theta) = \int_{-t}^{0} \int_{-1}^{1} H(x,z) \exp.(kz-ik\, x \cos\theta)dxdz, \tag{3.2}$$

where $H(x,z)$ means the breadth of the ship.

First, we will show a type of wave-free distribution, when H has a form

$$H(x,z) = T(z)H(x) . \tag{3.3}$$

Then, the function in (3.2) reduces to

$$F(k,\theta) = I(k)f(k \cos\theta) , \tag{3.4}$$

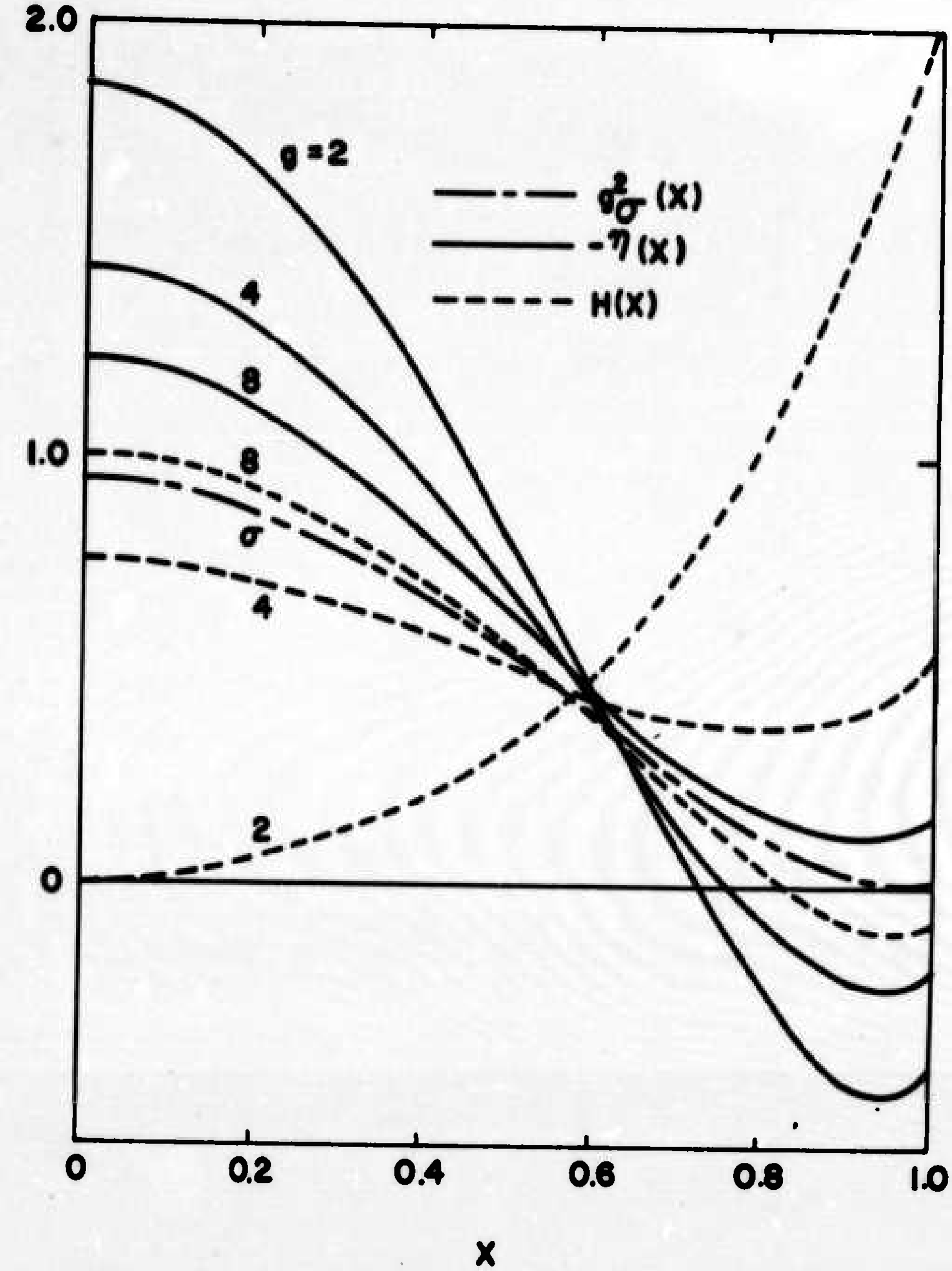
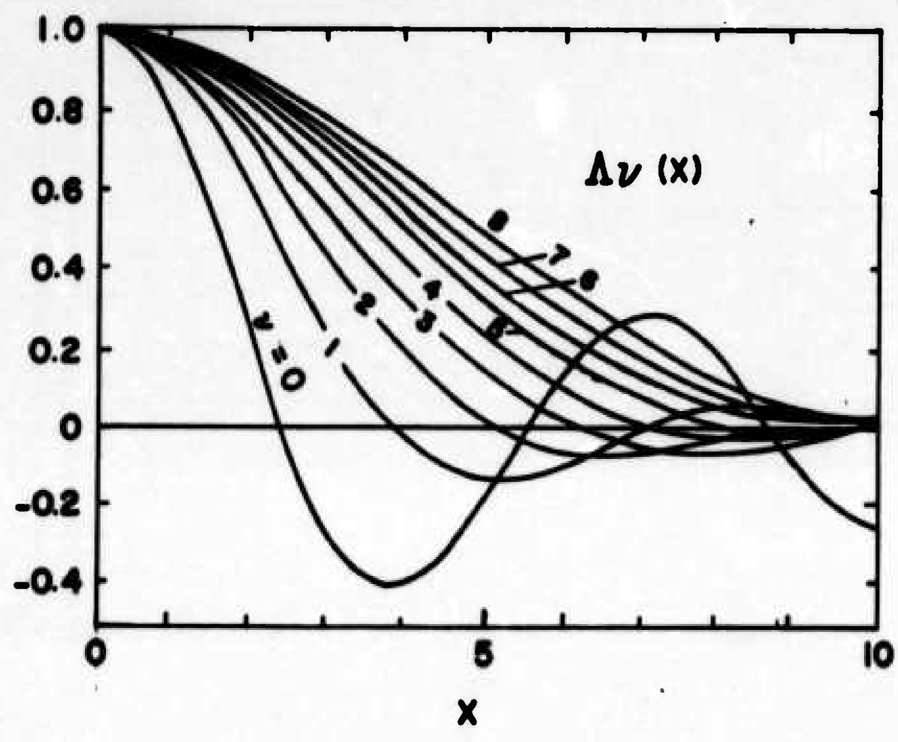


Figure 1.



$$\Lambda_\nu(X) = \frac{\Gamma(\nu+1)}{(gx/2)^\nu} J_\nu(gx)$$

Figure 2.

where

$$I(k) = \int_{-t}^{0} T(z) \exp(kz)\, dz, \tag{3.5}$$

and

$$f(p) = \int_{-1}^{1} H(x) \exp(-ipx)\, dx, \tag{3.6}$$

Now, if

$$f(p) = 0, \quad \text{for } p = g\sec\theta > g, \tag{3.7}$$

then we clearly have a type of wave-free distribution.

This condition (3.7) is fulfilled if $H(x)$ is represented as

$$H(x) = \int_{-g}^{g} f(p) \exp(ipx)\, dp, \tag{3.8}$$

and so the range of distribution tends to infinity and generally it takes negative value in some portions.

For example, let us put

$$f(-g\cos\theta) = \cos n\theta/(g\sin\theta) \quad , \tag{3.9}$$

then we have

$$H(x) = 2\pi\,(-i)^{n} J_n(gx)\ , \tag{3.10}$$

that is, they are represented by Bessel functions.

For another example, let us put

$$f(-g\cos\theta) = (2\sqrt{\pi}/g)[\Gamma(\nu+1)/\Gamma(\nu+1/2)]\, \sin^{\nu-1}\theta,\ \nu > 0 \tag{3.11}$$

then, we have

$$H(x) = \Lambda_\nu(x) = \Gamma(\nu+1)\, J_\nu(gx)/(gx/2)^{\nu}\ , \tag{3.12}$$

These curves are shown in Figure 2 and we see that these, especially for small ν, tend rapidly to zero by departing the origin, so that we may obtain the distribution with small wave-resistance by cutting out both ends.

4. Three Dimensional Problem II[6,7]

If the distribution is confined to some finite area, then we may treat it the same way as in paragraph 2.

Introduce an auxiliary function by the next differential equation.

$$[\partial/\partial z - (1/g)\, \partial^2/\partial x^2]\, \sigma(x,z) = H(x,z) \quad . \qquad (4.1)$$

Then, putting this into (3.2) and integrating by parts, we have

$$F(g \sec^2\theta,\theta) = \int_{-1}^{1} [\sigma(x,z) \exp(gz \sec^2\theta - igx \sec\theta)]_{z=-t}^{0} dx$$

$$-\int_{-t}^{0} [(\sigma + ig \sec\theta\, \partial\sigma/\partial x) \exp(gz \sec^2\theta - igx \sec\theta)]_{x=-1}^{1} dz \ . \qquad (4.2)$$

Hence, if we have

$$\sigma(x,0) = \sigma(x,-t) = \sigma(\pm 1,z) = \partial\sigma/\partial x(\pm 1,z) = 0, \quad (4.3)$$

then F vanishes and we have a wave-free distribution. However, integrating (4.1) and putting (4.3), we have

$$\int_{-t}^{0}\int_{-1}^{1} H(x,z)\, dxdz = 0, \qquad (4.4)$$

namely, the total displacement of this distribution vanishes.

This is a different result from the above obtained, and there is no more interest with this for the object only to obtain the wave-free ship, but this distribution suggests a possibility to deform a ship shape in a certain arbitrary degree without change of the wave-resistance.

Namely, even if we add to or subtract from the given distribution this wave free one multiplied by an arbitrary constant, the wave-resistance is kept unchanged as easily seen.

We will call this method the invariant deformation. Considering in this way, (4.4) is an important property of such deformation, and that we have easily

$$\int_{-t}^{0}\int_{-1}^{1} xH(x,z)\, dxdz = 0 \ , \qquad (4.5)$$

which says that the center of buoyancy does not also change by such deformations.

For example, consider the function of the next type

$$\sigma(x,z) = X(x)T(\zeta), \quad \zeta = (z+t)/t, \tag{4.6}$$

and put

$$\begin{aligned} X_0(x) &= \xi^4(1-\xi)^2, \\ X_1(x) &= 1-4\xi^3 + 3\xi^4, \\ X_2(x) &= 1-10\xi^3 + 15\xi^4 - 6\xi^5, \end{aligned} \tag{4.7}$$

where $\xi = (x-b)/(1-b)$ for $1 > x > b$ and $\xi = (-a-x)/(1-a)$ for $-1 < x < -a$, a and b are arbitrary constants, and let $T_1(\zeta)$ be the function of which derivative is composed of three segments and has the values

$$T_1'(0) = 5, \quad T_1'(0.2) = 1 \quad \text{and} \quad T_1'(0.4) = T_1'(1) = -1 \tag{4.8}$$

and

$$T_2(\zeta) = \zeta(1 - \zeta) . \tag{4.8a}$$

Example 1.

At the speed Fr. = 0.212, put

$$\begin{aligned} X(x) &= X_0(x) \quad \text{for} \quad 1 > x > b = 0.5, \\ &= 0 \qquad\ \ \text{for} \quad b > x > -1 , \\ T(\zeta) &= T_1(\zeta) . \end{aligned} \tag{4.9}$$

From these functions, compute $H(x,z)$ by (4.1) and subtract it from the off-sets of the ship showed by full lines in Figure 3, which is of an oil tanker. Then, the bulbous bow may become a conventional one which is shown by dotted lines. Here we replace the section form at F.P. by the chain line which is represented by $T_1(\zeta)$, merely because the full line is too complex to represent by a simple formula.

Example 2.

At the speed Fr. = 0.184 and of another oil tanker shown by full lines in Figure 4, put

Figure 3.

M. S. No. 1342
C_b = .80
L/B = 7.2
B/d = 2.46
L/d = 17.7

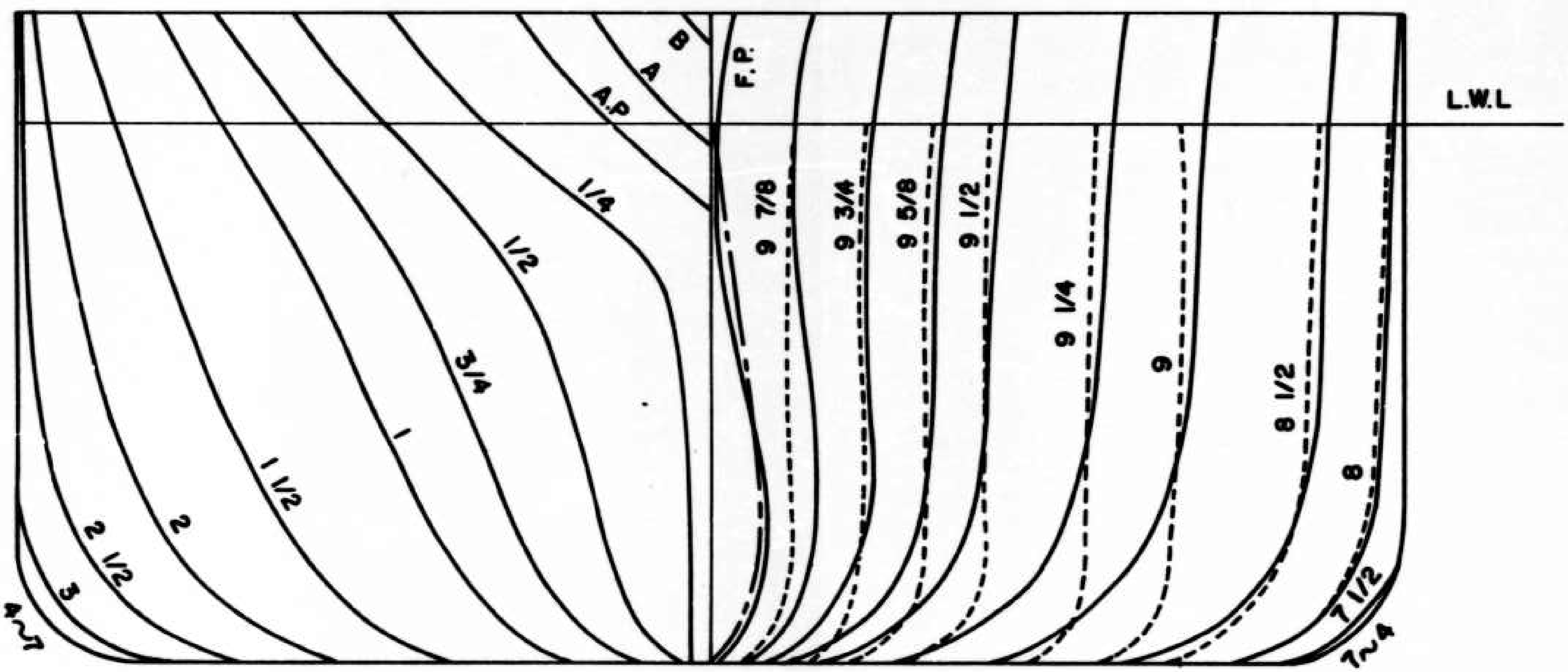
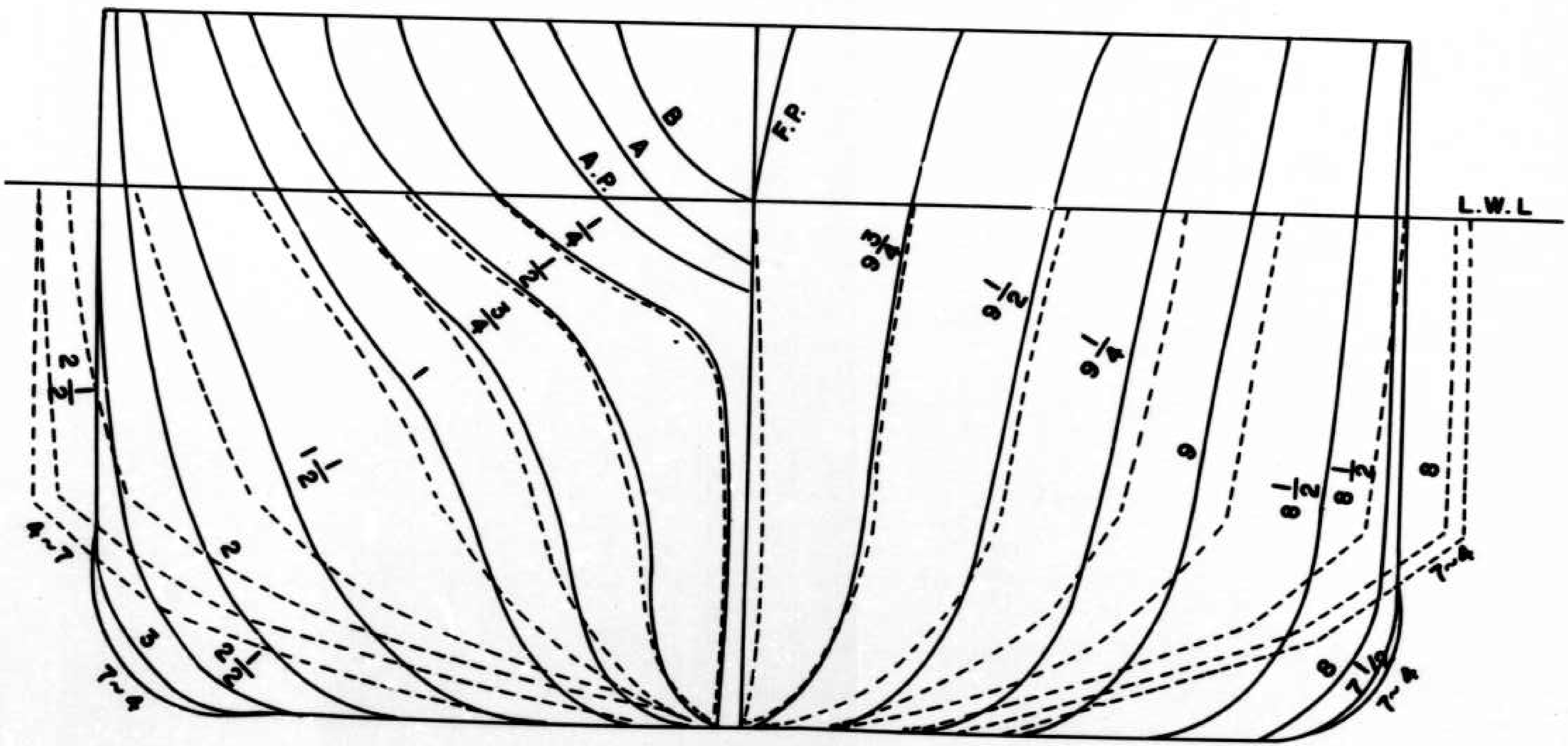


Figure 4.

Figure 4.

M. S. No. 1321
$C_b = .80$
$L/B = 7.34$
$B/d = 2.46$
$L/d = 18.08$



$$
\begin{aligned}
X(x) &= X_1(x) && \text{for} \quad 1 > x > b = 0.5, \\
&= 1 && \text{for} \quad b > x > -a = -0.5, \\
&= X_2(x) && \text{for} \quad -a > x > -1 \; , \\
T(\zeta) &= T_1(\zeta) \; ,
\end{aligned}
\qquad (4.10)
$$

and proceed as the above, we have dotted lines of which breadth is broadened about 10%.

Example 3.

For the same speed and the same ship as in Example 2, put

$$
\begin{aligned}
X(x) &= X_2(x) && \text{for} \quad 1 \geq x \geq b = 0.5, \\
&= 1 && \text{for} \quad b > x > -a = -0.5, \\
&= X_2(x) && \text{for} \quad -a > x > -1 \; ,
\end{aligned}
\qquad (4.11)
$$

and $$T(\zeta) = T_2(\zeta) \; .$$

Then, as we see in Figure 5 by dotted lines, we may have the ship shape with inclined side shell.

In the latter two examples, the sectional area does not change so much that we may understand the well-known practice which says that the residual resistance of ships does not change so much in such cases. Namely, this method may propose a theoretical foundation to the problem of deforming the ship shape without sacrifice of the resistance so that we may easily find a suitable solution to the demand of the ship designer.

5. Transverse Wave-Free Distribution[7]

The method used in paragraph 2 is also applicable to making the wave element vanish in three dimensional case.

For example, put $\Theta = 0$ and $k = g$ in (3.2), that is, consider the transverse wave only and the distribution of the type as (3.3), then

$$F(g,0) = \int_{-t}^{0} T(z)\exp(gz)dz \cdot \int_{-1}^{1} H(x)\exp(-igx)dx. \qquad (5.1)$$

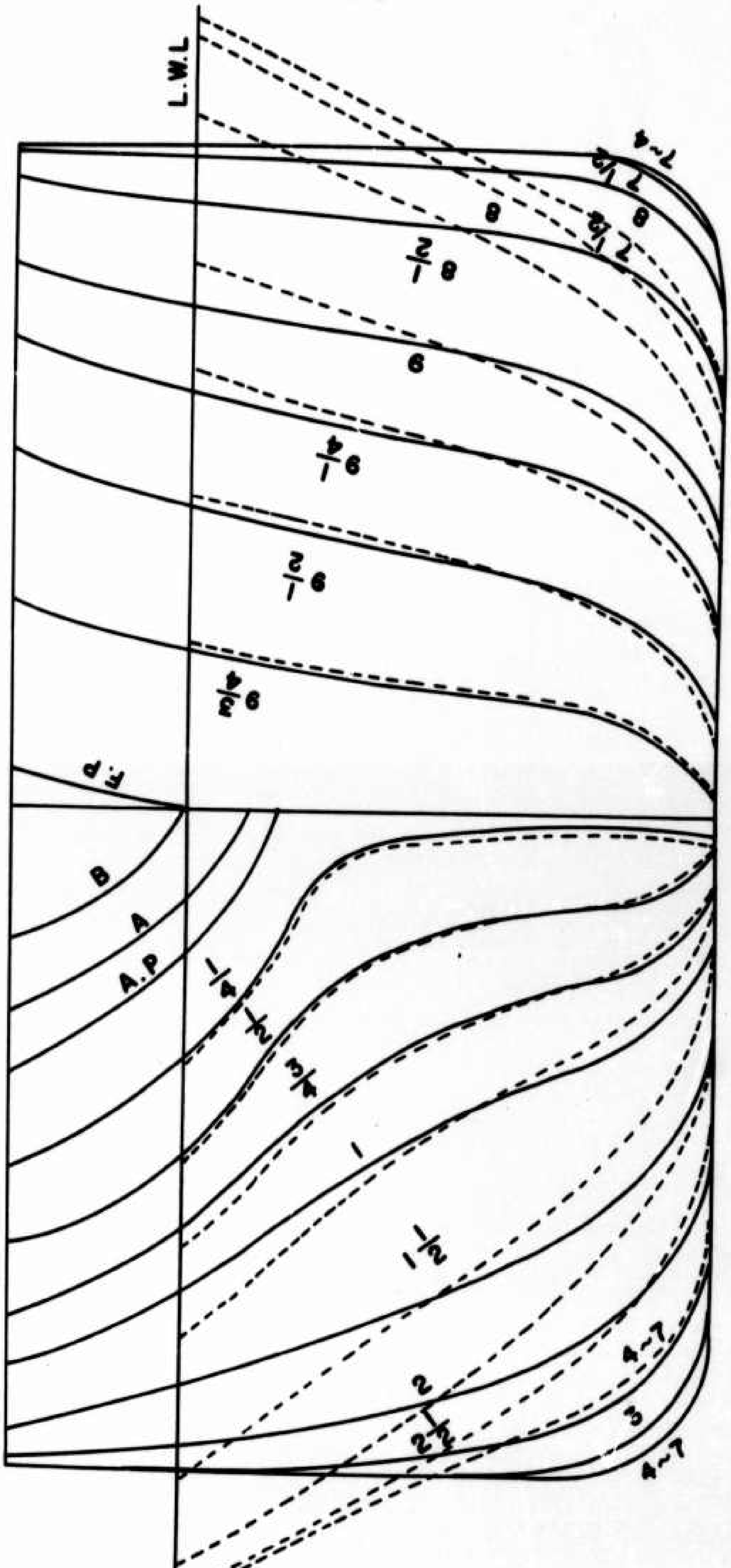


Figure 5. M. S. No. 1321.

Figure 5. M. S. No. 1321.

The integral in x is the same form as in (2.2), so that we may have a wave-free distribution by introducing the function σ defined by (2.4) and having the boundary values (2.7). We have seen an example of such distribution on Figure 1, and this may be a very simple method to obtain a ship form with small wave-resistance.

Let us see one more example. Put

$$H(x) = a_0[M(x) + (1/g^2)M''(x)] + a_2[N(x) + (1/g^2)N''(x)], \quad (5.2)$$

with

$$M(x) = (1-x^2)^2 \quad \text{and} \quad N(x) = x^2(1-x^2)^2 , \quad (5.3)$$

where a_0 and a_2 are determined so as to $H(0) = 1/0.6$.

Then, we have curves in Figure 6 compared with the distributions of the minimum wave-resistance which are drawn by dotted lines. These curves show the similarity between both groups, so that we may suppose the wave-resistance of the present one's comparatively small.

Lastly, it is noteworthy that we may have also a transverse wave-free distribution by introducing the next function, considering in (5.1),

$$(d/dz + g)\,\mu\,(z) = T(z) ,$$

with $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (5.4)

$$\mu(0) = \mu(-t) = 0 .$$

6. Conclusion

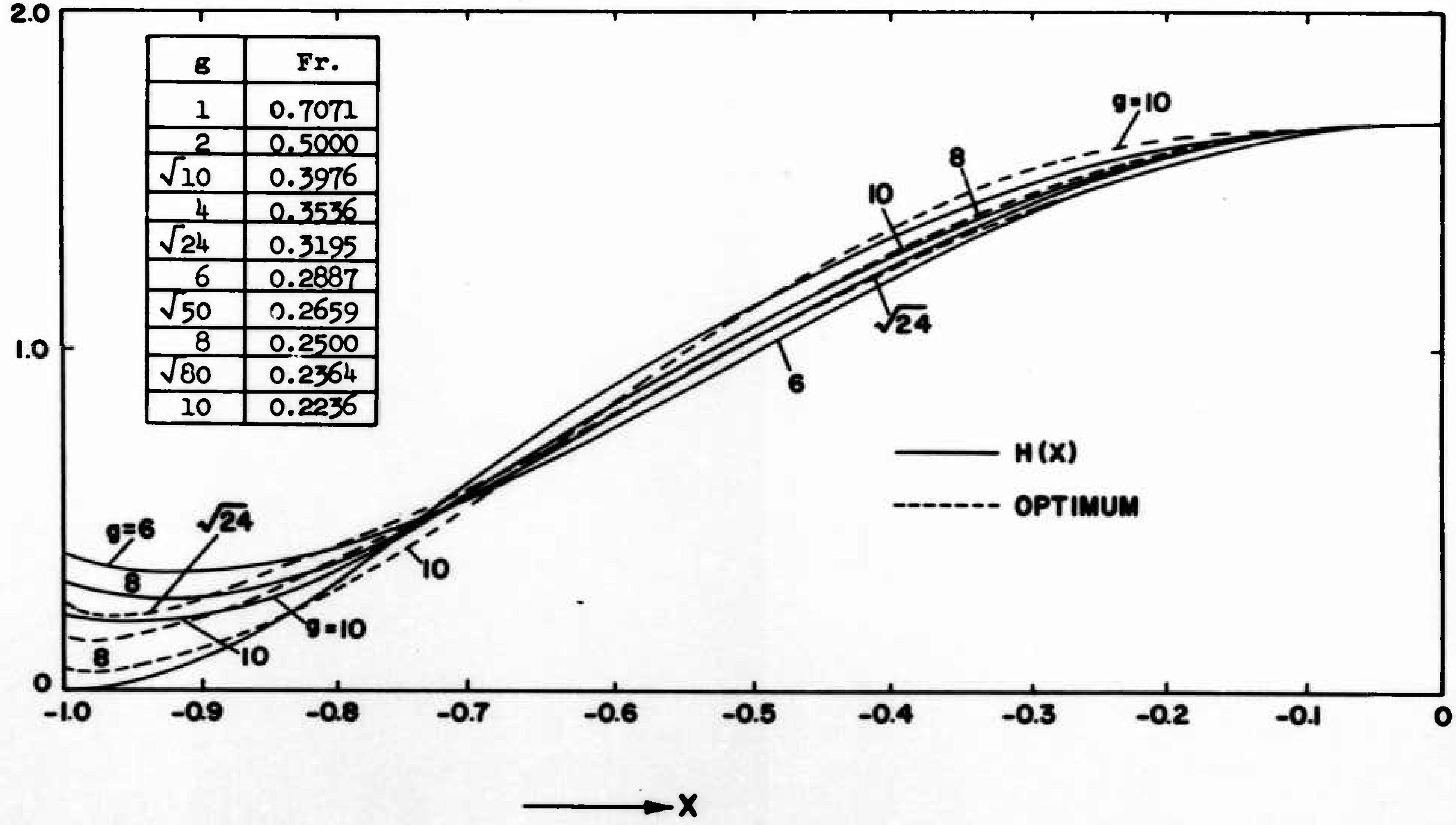
We may summarize the conclusions as follows:

i) The two dimensional wave-free distributions are obtained and applied to obtain ship water lines with small wave-resistance.

ii) The three dimensional ones distributed over an infinitely long range are obtained and may be useful to obtain such water lines as the above.

iii) The three dimensional ones over a finite area are obtained too, but have no displacement and longitudinal moment.

Figure 6. H(x), δ = 0.6.



g	Fr.
1	0.7071
2	0.5000
$\sqrt{10}$	0.3976
4	0.3536
$\sqrt{24}$	0.3195
6	0.2887
$\sqrt{50}$	0.2659
8	0.2500
$\sqrt{80}$	0.2364
10	0.2236

Figure 6. H(x), δ = 0.6.

However, they are useful in the sense that we may deform a ship shape without change of the wave-making resistance.

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